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ON THE GENERAL NOTION OF FULLY NONLINEAR SECOND-ORDER ELLIPTIC EQUATIONS

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ABSTRACT. The general notion of fully nonlinear second-order elliptic equation is given. Its relation to so-called Bellman equations is investigated. A general existence theorem for the equations like $P_m(u_{x^ix^j}) = \sum_{k=0}^{m-1} c_k(x)P_k(u_{x^ix^j})$ is obtained as an example of an application of the general notion of fully nonlinear elliptic equations.

We will be dealing with the following question: Given an equation

$$F(u_{x^ix^j}(x), u_{x^i}(x), u(x), x) = 0$$

in a domain $D \subset \mathbb{R}^d$, under what conditions and in what class of twice continuously differentiable functions u do we call it an elliptic equation?

It may look strange that we address such a question. Indeed, there are even books [9], [17] and many articles about the general theory of fully nonlinear elliptic equations. Therefore, very general results are available in the theory. However, on the other hand, it turns out that if an unexperienced reader meets a fully nonlinear second-order partial differential equation in his investigations and tries to get any information about its solvability from the literature, then almost certainly he fails to find what he needs, unless he considers an equation that is exactly one which had already been treated. The point is that in the general theory we consider nonlinear equations only of a special type, say, such that for any $x \in D$, $\xi \in \mathbb{R}^d$, u_{ij}, u_i, $u \in R$

$$\delta|\xi|^2 \le F(u_{ij} + \xi^i\xi^j, z) - F(u_{ij}, z) \le \delta^{-1}|\xi|^2,$$

where $z = (u_i, u, x)$ and δ is a positive constant. Obviously, even for the simplest Monge-Ampère equation, when $F = \det(u_{ij}) - f(x)$, conditions of this type are not satisfied (for instance, for $d \ge 2$ the function $\det(u_{ij})$ grows much faster than linearly). Nevertheless, the general theory applies to this and many other special equations, and the reason for this is that *in an appropriate class of functions they can be reduced* to those considered in the general theory. We apply some techniques to include into our theory concrete equations such as the Monge-Ampère equation (real or complex) or more general Weingarten equations. These techniques are different in different cases, and by answering our main question here we want to show, in particular, *what should be done for*

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an arbitrary equation in order to understand if the general theory applies to it, and if it does, through what part of the theory to look.

In many respects our approach to the definition is very close to the one from [5]. But since the main purpose there was to prove existence theorems and by methods known at that time, the class of equations from [5] does not include many degenerate nonlinear equations. For example, the degenerate Monge-Ampère equation $\det(u_{x^ix^j}) = 0$ does not fall into the scheme from [5]. In a sense, our conditions are easier to verify, and they do not exclude this example.

It should be noted that our definitions are adapted for the *existing theory* and that so far, if an equation does not fall into our scheme, there is no way to investigate its solvability. But it is conceivable that other theories will give rise to other definitions.

The article is organized as follows. In Section 1 we discuss different approaches to the notion, present main features of our approach, and state Theorem 1.1 about an analog of the Monge-Ampère equation which we prove in Section 5 simply by examining the functions defining the equation and by referring to our general results. Section 2 contains main definitions, based on the notion of an *elliptic branch* of the equation, and simple examples of nonlinear equations illustrating the definitions. In Section 3 we write down elliptic branches in the form of usual equations, and in Section 4 we consider the most well-understood case when an elliptic branch is defined by a convex domain and can be described by the Bellman equation. There we also prove Theorem 1.1 in the case when D is strictly convex and l_k are constant. The proof is based on some general results from Section 6, allowing one to recognize when elliptic branches of nonlinear equations are defined by convex domains.

In a sense, Sections 1 through 4 give the necessary tools to understand if the general theory applies to a given fully nonlinear equation. We consider there, so to speak, only the principal part of the equation without independent variables entering explicitly.

Section 5 deals with the question of *how to apply* the general theory in some cases when independent variables do enter the equation. Specifically, we show how to use properties of an elliptic branch in order to understand if the function defining its usual form possesses properties required in the general theory. The trial example here is Theorem 1.1. In total we give proofs of three versions of this theorem, gradually increasing the generality and checking only one part of the conditions of a general theorem about Bellman equations at a time. The purpose of this is to show that whenever one needs a stronger result about an elliptic nonlinear PDE, one should investigate deeper properties of functions *defining* the equation, that is, make a kind of analytical work which has nothing at all to do with the PDE theory. Naturally, the proofs of these versions rely upon some analytical facts, specifically, concerning hyperbolic polynomials, and they are proved in the last Section 6 of the article.

1. Preliminaries

Naturally, the type of equation should be defined only by the dependence of F on $u_{x^ix^j}$; that is, we will call our equation elliptic if for any $p \in \mathbb{R}^d$, $y \in D$, and $z \in R$ the following equation in D is elliptic: $F(u_{x^ix^j}(x), p, z, y) = 0$. Therefore, let us concentrate on the case when F depends only on the matrix

of second-order derivatives of u; in other words, we will consider the equation

$$F(u_{x^i x^j}(x)) = 0. \tag{1.1}$$

We assume of course that the set

$$\Gamma := \{(v_{ij}) : v_{ij} = v_{ji}\,,\ i\,,\ j = 1\,,\ \dots\,,\ d\,,\ F(v_{ij}) = 0\}$$

is not empty.

At first sight it seems natural that equation (1.1) should be called elliptic only if equations $F(u_{x^i x^j}(x)) + c = 0$ are elliptic for any constant c. Indeed, this is the case at least for linear equations. Nevertheless, the *experience* of dealing with nonlinear elliptic equations shows that, actually, we should not try to keep this property. For instance, in the future we will see that the equation

$$2u_{x^1x^1}^2 + 5u_{x^1x^1}u_{x^2x^2} + 2u_{x^2x^2}^2 = 1 \tag{1.2}$$

is an elliptic and even uniformly elliptic equation and the equation

$$2u_{x^1x^1}^2 + 5u_{x^1x^1}u_{x^2x^2} + 2u_{x^2x^2}^2 = -1 \tag{1.3}$$

does not behave as an elliptic equation at all (see Example 2.21 below). This means that we should investigate each individual equation like (1.1) separately.

Usually in the literature on nonlinear elliptic equations (see, for instance, [6], [3], [4], [9], [17]) equation (1.1) is called elliptic if the matrix $\partial F/\partial u_{ij}$ is nonnegative (or nonpositive) for *all arguments*. Of course, this excludes at once even the simplest Monge-Ampère equation and forces the reader who has come up with a concrete equation to look where applications are considered.

An attempt to give a better definition is made in [2] where the equation is called elliptic *on a given solution* u if at any point $x \in D$ the matrix with entries $\partial F/\partial u_{ij}(u_{x^i x^j}(x))$ is nonnegative (or nonpositive). After that equation (1.1) is called elliptic in the given class $\mathscr{C}$ (say, $\mathscr{C} = C^2(D)$) of functions if it is elliptic on any (if there is any) solution $u \in \mathscr{C}$. It is worth noting that only in rare cases we can take $\mathscr{C} = C^2(D)$ in this definition. For example, for $d = 3$ the function $u(x) = (x^1)^2 - (x^2)^2 - (x^3)^2$ is a solution of the simplest Monge-Ampère equation $\det(u_{x^i x^j}) = 8$, which as we will see later should be called elliptic, but the matrix $(\partial F/\partial u_{ij}(u_{x^i x^j}))$ is indefinite.

This definition has several weak points. For instance, as easy to check equations (1.2), (1.3) are then both elliptic in the same class of functions $\mathscr{C}$ defined as the set of all functions for which $\Delta u \geq 1/\sqrt{18}$. Furthermore, somehow we should know a priori in what class of functions (say, convex, plurisubharmonic, ...) the given equation can and should be considered. From the point of view of this definition the research carried out in [13] appears somewhat mysterious and looks like the author found a right class of functions by chance. (An unsuccessful attempt to apply the above definition in the case of equation (1.3) is given in Example 2.21 below).

It is also worth noting that the objective is not only to give a definition of a nonlinear elliptic equation but also to find such a definition which could do the job. For instance, we are usually interested in the uniqueness of solutions, and we usually prove it via the maximum principle. In other words, if we are given two solutions u_1, u_2 of equation (1.1), then by proceeding as usual (cf.,

for instance, [6, Chapter 4, Section 6.2]) for $v = u_1 - u_2$ we write

$$0 = F(u_{1x^ix^j}(x)) - F(u_{2x^ix^j}(x)) = a^{ij}(x)v_{x^ix^j}(x),$$

where

$$a^{ij} = \int_0^1 \frac{\partial F}{\partial u_{ij}}(tu_{1x^ix^j} + (1-t)u_{2x^ix^j})\,dt,$$

and we expect the matrix $a = (a^{ij})$ to be positive or negative. If we assume the above definition from [2], then we know that the matrices $\partial F/\partial u_{ij}$ are say, positive *on* u_1 *and on* u_2, but generally speaking, $tu_1 + (1-t)u_2$ is not a solution and we do not know anything about the definiteness of a. Actually, it may even happen that for one function F the matrix a is always positive, and for another function F, defining an equation *equivalent* to initial equation (1.1), the corresponding matrix a is neither positive nor negative. The point is that we can arbitrarily modify the function F outside the set Γ, the only set where some properties of F are given so far. By the way, this possibility of modification of a nonlinear equation is the main reason for the radical difference between linear and nonlinear equations, since for the linear case the set Γ is a hyperplane in the linear space

$$\mathbb{S}^d = \mathbb{R}^k \cap \{(u_{ij}) : u_{ij} = u_{ji},\ i, j = 1, \dots, d\},$$

where $k = d^2$, and there are not so many ways to represent a hyperplane as a null set of a *linear* function.

One way to overcome the last difficulty is to accept the notion of elliptic convexity of F from [2], that is, to consider only F such that for any two solutions (from the class $\mathscr{C}$) the matrix a is positive. In this system of notions, given an equation, to decide if it is a "legal" elliptic equation, first we should guess in what class of functions we will look for solutions and then modify (if it is possible at all) the function F, without changing the equation, in order to replace it with an elliptically convex F.

Unfortunately, even after this other difficulties still remain. For instance, assume that at the very beginning we know the appropriate class of functions $\mathscr{C}$ and our F is elliptically convex in this class. Assume that we even obtained a priori estimates for solutions of the equation. The question of how to prove existence theorems arises.

Usually we introduce a parameter $t \in [0, 1]$ and try to find functions F_t continuous in t such that $F_1 = F$ and F_0 defines an equation for which everything is known. After this we try to prove the same a priori estimates for solutions, *belonging to the same class* $\mathscr{C}$, of equations corresponding to F_t for all $t \in [0, 1]$, and then apply some topological methods to get the solvability of the equation $F_t(u_{x^ix^j}) = 0$ for $t = 1$ from its solvability for $t = 0$. But on this way, in all interesting cases, we cannot afford to take F_0 linear since usually solutions of linear equations have no reason to belong to $\mathscr{C}$. For instance, for the Monge-Ampère equation $\det(u_{x^ix^j}) = 1$ in a strictly convex domain D with boundary data on ∂D, one of the right classes of solutions is the class of all convex functions. At the same time there is no linear equations for which all solutions with different boundary data are convex.

In a way, this cuts us off from the linear theory and raises the obscure problem of finding a "model" nonlinear function F_0 for any particular F. For professionals in the field this problem is not too hard, and many authors prefer to use model equations while treating concrete equations (see, for instance, [2], [3], [13]), but for a "ready-to-use" theory this "cut off" is highly undesirable since applications may advance equations different from those which have already been investigated. However, in the above system of notions we cannot avoid this difficulty unless we can either understand how to make the continuation with respect to the parameter t in the situation when the set $\mathscr{E}_t$ of solutions is evolving with t or "hide" the set $\mathscr{E}$ by finding a function $\tilde{F}$ such that *any solution of* (1.1) *of class* $\mathscr{E}$ is a solution to the equation

$$\tilde{F}(u_{x^ix^j}(x)) = 0, \tag{1.4}$$

and vice versa, *any* solution of (1.4) is a solution of (1.1) and belongs to $\mathscr{E}$. In this paper we shall explore the latter possibility.

We shall present a different approach to the notion of nonlinear elliptic equation. We shall give a method to decide if a given nonlinear equation is an elliptic one by *looking only at the equation.* After this we give the notion of admissible solutions of the equation and then discuss the possibility of rewriting the equation with the help of elliptically convex functions F.

The most important concept in this approach is the notion of an admissible solution which shows the right class of functions in which to look for solutions. This notion is based on the notion of *elliptic branches* of the given equation, which turns out to be meaningful even for viscosity solutions of the *first*-order nonlinear equations (see Remark 2.25 below).

It is worth noting that in all cases in the literature our class of admissible solutions coincides with the known ones.

As an example of the application of our general setup we prove the following new result.

Theorem 1.1. *Let $d \geq 2$, m be an integer, $1 \leq m \leq d$, and $P_m(\lambda) = P_{m,d}(\lambda)$ be the mth elementary symmetric polynomial of the variables $\lambda = (\lambda^1, \dots, \lambda^d)$. Define $P_0 = P_{0,d} \equiv 1$. Let D be a bounded domain of class C^4 with connected boundary ∂D and such that at any point of ∂D we have*

$$P_{m-1,d-1}(\kappa^1, \dots, \kappa^{d-1}) > 0,$$

where $\kappa^1, \dots, \kappa^{d-1}$ are the principal curvatures of ∂D at this point evaluated with respect to the interior normal to ∂D (so that for spheres all of them are positive). Let $l_0, \dots, l_{m-1} \in C^2(\mathbb{R}^d)$, and let $\phi \in C^4(\mathbb{R}^d)$. For any $w \in \mathbb{S}^d$ define $\lambda(w)$ as a vector of eigenvalues of w ordered arbitrarily and define $P_m(w) = P_m(\lambda(w))$. Then the equation

$$P_m(u_{x^ix^j}) = \sum_{k=0}^{m-1} (l_k^+)^{m-k+1}(x) P_k(u_{x^ix^j}) \quad (a.e.) \text{ in } D \tag{1.5}$$

with the boundary condition $u = \phi$ on ∂D has a unique solution $u \in C^{1,1}(\bar{D})$ characterized by the additional property that

$$P_m(u_{x^ix^j} + t\delta_{ij}) > 0,$$

$$(1.6)\qquad P_m(u_{x^ix^j}+t\delta_{ij}) > \sum_{k=0}^{m-1}(l_k^+)^{m-k+1}(x)P_k(u_{x^ix^j}+t\delta_{ij})\ (a.e.)\ in\ D$$

for any $t > 0$. *Moreover, if* $\sum_k l_k^+ > 0$ *in* $\bar{D}$, *then* $u \in C^{2+\alpha}(\bar{D})$ *for an* $\alpha \in (0, 1)$.

Remark 1.2. If $D = \{|x| < 1\}$ and we consider the equation $P_d(u_{x^ix^j}) = P_{d-1}(u_{x^ix^j})$ with zero boundary condition, then one of the possible solutions is identically zero. It is important to stress that it is not the one about which we are talking in the theorem. From the point of view of conditions (1.6) the admissible solution will be $u = -c(1-|x|^2)$, where $2c = P_{d-1}(\delta_{ij})/P_d(\delta_{ij})$.

We could also consider many other equations involving P_m (cf. Remark 6.7 below). The choice of equation (1.5) is caused by the popularity of such equations in geometry. Equations of type (1.5) with $m = d$ had been first considered in [2, Section 20] where the existence of a special kind of generalized solution was established under assumptions *different* from ours. Results and comments about the particular case of (1.5) when $l_k \equiv 0$ for $k = 1, \dots, m-1$ can be found in [5], [14], [20] (in [17] the reader can also find the case when $l_k \equiv 0$ only for $k = 2, \dots, m-1$). It is possible that the powers $m-k+1$ in (1.5) can be replaced by $m-k$, at least this is the case when $l_k \not\equiv 0$ for one value of k only (see Remark 5.14 below; for the case when this $k = 0$ see also, for instance, [20]). In Remark 5.17 we discuss the issue of better regularity of solutions.

2. Definitions and examples

Our point of view is based on the observation that every individual equation (1.1) means and means only that for any $x \in D$

$$(2.1)\qquad (u_{x^ix^j}(x)) \in \Gamma$$

(recall that $\Gamma := \{(v_{ij}) : v_{ij} = v_{ji},\ i, j = 1, \dots, d,\ F(v_{ij}) = 0\}$). This point of view allows us to concentrate on properties of the set Γ rather than occasional properties of the numerous functions which define the same set Γ. Only properties of the set Γ define the type of the equation.

Of course, we assume that F is at least a continuous function, which implies that Γ is a closed set of the space $\mathbb{S}^d$. We also keep the assumption from Section 1 that $\Gamma \neq \emptyset$. We have already treated the set of all symmetric $d \times d$ matrices as a subspace $\mathbb{S}^d$ in the Euclidean space $\mathbb{R}^k$. In the sequel we need to use the Euclidean norm $||\cdot||$ in this space. Note that for $w \in \mathbb{R}^k$ we have $||w||^2 := \operatorname{tr} ww^*$. To finish with notation, we denote by I the unit $d \times d$ matrix and define $\mathscr{M}_+$ ($\mathscr{M}_+^o$) as the set of all nonnegative (respectively, strictly positive) symmetric $d \times d$ matrices.

Definition 2.1. We say that a nonempty open (in $\mathbb{S}^d$) set $\Theta \neq \mathbb{S}^d$ is a (positive) *elliptic* set if

(a) $\Theta = \bar{\Theta} \setminus \partial(\bar{\Theta})$,

(b) for any $(u_{ij}) \in \partial\Theta$, $\xi \in \mathbb{R}^d$ it holds that $(u_{ij} + \xi^i\xi^j) \in \bar{\Theta}$.

Definition 2.2. We say that equation (1.1) (or, more generally, equation (2.1) with any nonempty closed Γ) is an *elliptic* equation if there is an elliptic set Θ

such that $\partial\Theta \subset \Gamma$. In this case we call the equation

$$(u_{x^i x^j}(x)) \in \partial\Theta, \quad x \in D, \tag{2.2}$$

an *elliptic branch of equation* (1.1) (*or* (2.1)) *defined by* Θ.

Definition 2.3. We say that an elliptic set Θ is *quasi-nondegenerate* if for any $(u_{ij}) \in \partial\Theta$, $\xi \in \mathbb{R}^d \setminus \{0\}$ we have $(u_{ij} + \xi^i\xi^j) \in \Theta$.

Given a number $\delta > 0$, we call an elliptic set Θ δ*-nondegenerate* (or *uniformly elliptic*) if for any $w \in \partial\Theta$, $\xi \in \mathbb{R}^d$ we have

$$\operatorname{dist}(w + \xi\xi^*, \partial\Theta) \geq \delta|\xi|^2.$$

If equation (2.2) is an elliptic branch of (1.1) (or (2.1)) and Θ is quasi-nondegenerate (δ-nondegenerate, uniformly elliptic), we call this branch and equation (1.1) (or (2.1)) quasi-nondegenerate (respectively, δ-nondegenerate, uniformly elliptic) (cf. Remark 2.5 below).

Remark 2.4. If Θ is an elliptic set and $\xi \in \mathbb{R}^d$, then $w + \xi\xi^* \in \bar{\Theta}$ not only for $w \in \partial\Theta$ but also for $w \in \Theta$, and in this case we even have that $w + \xi\xi^* \in \Theta$.

Indeed, for $w \in \Theta$ the open set $T := \{t > 0 : w + t\xi\xi^* \notin \bar{\Theta}\}$ is empty since if not, then for any interval (a, b) composing T we would have $w_1 := w + a\xi\xi^* \in \partial\Theta$ and $w_1 + \varepsilon\xi\xi^* \notin \bar{\Theta}$ if $0 < \varepsilon < b - a$. Therefore, $w + \xi\xi^* \in \bar{\Theta}$. Actually, $w + \xi\xi^* \notin \partial\Theta$ because we can move the point w a little bit in any direction without violating the property $w + \xi\xi^* \in \bar{\Theta}$.

Moreover, any matrix $v \in \mathscr{M}_+$ can be represented as $\xi_1\xi_1^* + \cdots + \xi_d\xi_d^*$ with $\xi_i \in \mathbb{R}^d$, and therefore, from the above it follows that if $w \in \bar{\Theta}$, then $w + v \in \bar{\Theta}$ for any such v. Since the set $\mathscr{M}_+^o$ is an open set in $\mathbb{S}^k$ and $\Theta = \bar{\Theta} \setminus \partial\bar{\Theta}$, it follows that for any $w \in \bar{\Theta}$, $v \in \mathscr{M}_+^o$ we have $w + v \in \Theta$. In particular, Θ is necessarily connected.

Furthermore, for any $v \in \mathscr{M}_+^o$ there is $t_0 \geq 0$ such that $tv \in \Theta$ for any $t > t_0$. Indeed, it suffices to take a $w \in \Theta$ and to notice that for sufficiently large t we have $v_1 = tv - w \in \mathscr{M}_+^o$.

Remark 2.5. We have just seen that if Θ is an elliptic set, then $\{w + v : w \in \partial\Theta, v \in \mathscr{M}_+^o\} \subset \Theta$. Actually, there is an equality here, and moreover,

$$\Theta = \{w + tI : w \in \partial\Theta, t > 0\}.$$

Indeed, if for a $u \in \Theta$ the half line with the equation $u - tI$, $t > 0$, does not intersect $\partial\Theta$, then it lies in Θ, and the whole set $\{u - tI + v : t > 0, v \in \mathscr{M}_+^o\}$ lies in Θ. But this is impossible since the former set coincides with $\mathbb{S}^d$.

In particular, an elliptic set is uniquely defined by its boundary. Specifically, there is no ambiguity in our Definition 2.3 of quasi-nondegeneracy or δ-nondegeneracy of an elliptic branch. This issue arises because equation (2.2) is uniquely defined by $\partial\Theta$ rather than Θ.

Remark 2.6. Let Θ be an elliptic set. From Remark 2.4 and the fact that $w = (w - \xi\xi^*) + \xi\xi^*$ it follows that if $w \in \partial\Theta$ and $\xi \in \mathbb{R}^d$, then $w - \xi\xi^* \notin \Theta$. Furthermore, if Θ is quasi-nondegenerate and $\xi \neq 0$, then $w - \xi\xi^* \notin \partial\Theta$ and $w - \xi\xi^* \notin \bar{\Theta}$.

Moreover, if the set is δ-nondegenerate, then $\operatorname{dist}(w - \xi\xi^*, \partial\Theta) \geq \delta|\xi|^2$, since if it were not true, for a point $w_1 \in \partial\Theta$ we would have

$$\begin{aligned}\delta|\xi|^2 > \operatorname{dist}(w - \xi\xi^*, \partial\Theta) &= ||w_1 - (w - \xi\xi^*)|| \\ &= ||(w_1 + \xi\xi^*) - w|| \geq \operatorname{dist}(w_1 + \xi\xi^*, \partial\Theta).\end{aligned}$$

Remark 2.7. We can restate the conclusions of Remark 2.6 in the following way. Define negative elliptic sets by replacing $(u_{ij} + \xi^i\xi^j)$ in Definition 2.1 with $(u_{ij} - \xi^i\xi^j)$ and use the same substitution in the definition of quasi-nondegeneracy and δ-nondegeneracy of negative elliptic sets. Then Remark 2.6 says that if a set Θ is positive elliptic (and quasi-nondegenerate, δ-nondegenerate), then its "complement" $\hat{\Theta} = \mathbb{S}^d \setminus \bar{\Theta}$ is negative elliptic (and respectively, quasi nondegenerate, δ-nondegenerate) too. The converse is also true, and since $\partial\hat{\Theta} = \partial\Theta$, there is no need in appealing to negative elliptic sets in order to get more elliptic branches of a given nonlinear equation.

We also see that a set Θ is elliptic if and only if the set $-(\mathbb{S}^d \setminus \bar{\Theta})$ is elliptic.

Remark 2.8. For an elliptic set Θ equation (2.2) is, of course, an elliptic equation. In addition it has only one branch (coinciding with the equation). Indeed, for any elliptic set Θ_1 such that $\partial\Theta_1 \subset \partial\Theta$ by Remark 2.5 we have $\Theta_1 \subset \Theta$. But the set Θ is connected, and therefore, from the inclusion $\partial\Theta_1 \subset \partial\Theta$ we get that, actually, $\Theta_1 = \Theta$ and $\partial\Theta_1 = \partial\Theta$.

Specifically, from this argument we also see that elliptic branches have no proper elliptic subbranches.

Remark 2.9. Similarly to Remark 2.4 by using the notion of negative elliptic sets from Remark 2.7 we can convince ourselves that for any $v \in \mathscr{M}_+^o$ we have $-tv \notin \bar{\Theta}$ for all sufficiently large $t > 0$. In particular, now we see that for any $v \in \mathscr{M}_+^o$ there is a $t_0 \in (-\infty, \infty)$ such that $t_0 v \in \partial\Theta$. Since $tv \in \Theta$ for $t > t_0$ and $tv \notin \bar{\Theta}$ for $t < t_0$, this t_0 is unique.

If a set Θ is convex, there is a simple necessary and sufficient condition for its ellipticity (cf. [5]).

Theorem 2.10. *Let Θ be an open convex set in $\mathbb{S}^d$ such that $\Theta \neq \mathbb{S}^d$. Then it is an elliptic set if and only if for every $v \in \mathscr{M}_+^o$, $tv \in \Theta$ whenever $t > 0$ is large enough.*

Proof. The necessity follows from Remark 2.4. To prove the sufficiency of the condition take $w \in \partial\Theta$, $\varepsilon > 0$, $\xi \in \mathbb{R}^d$. Then for all large t we have $t(\varepsilon I + \xi\xi^*) \in \Theta$ and $w(1 - t^{-1}) + t^{-1}(t(\varepsilon I + \xi\xi^*)) \in \Theta$. When $t \to \infty$ this implies that $w + \varepsilon I + \xi\xi^* \in \bar{\Theta}$. At last, as $\varepsilon \downarrow 0$ we get $w + \xi\xi^* \in \bar{\Theta}$. The theorem is proved.

Remark 2.11. There is an obvious counterpart of Theorem 2.10 when the set $\mathbb{S}^d \setminus \Theta$ is convex. The reader can state and prove the corresponding assertion himself if he notices that for any function u satisfying (2.2) the function $v = -u$ satisfies the equation $(v_{x^ix^j}) \in \partial[-(\mathbb{S}^d \setminus \bar{\Theta})]$ and $-(\mathbb{S}^d \setminus \bar{\Theta})$ is elliptic whenever Θ is elliptic.

In terms of the initial function F Theorem 2.10 leads to the following result.

Theorem 2.12. *Let F be a continuous function on $\mathbb{S}^d$, and assume that for every $v_0 \in \mathscr{M}_+^o$ there is a neighborhood V of v_0 and a number t_0 such that we have $F(tv) > 0$ if $t \geq t_0$, $v \in V$. Assume that the open connected component Θ of the set $\{w : F(w) > 0\}$, such that $tI \in \Theta$ for all large t, is convex and $\Theta \neq \mathbb{S}^d$. Then equation* (1.1) *is elliptic and one of its elliptic branches is defined by Θ. Moreover, if there is no $w \in \partial\Theta$ such that $F(w + t\xi\xi^*) = 0$ for a $\xi \in \mathbb{R}^d \setminus \{0\}$ and all $t \geq 0$, then this branch and equation* (1.1) *are quasi nondegenerate.*

Proof. The last assertion is simply a specification of the definitions for the case of convex Θ defining an elliptic branch. In order to prove the first one we have to check that for every $v \in \mathscr{M}_+^o$, $tv \in \Theta$ whenever $t > 0$ is large enough. Fix a $v \in \mathscr{M}_+^o$ and let this be false. Then there is a sequence $t_n \to \infty$ such that $t_n v \notin \Theta$. We can assume that $t_n I \in \Theta$, and therefore, there exists $\varepsilon_n \in [0, 1]$ with the property $t_n(\varepsilon_n v + (1 - \varepsilon_n)I) \in \partial\Theta$. Of course, $F(t_n(\varepsilon_n v + (1 - \varepsilon_n)I)) = 0$. At last, without loss of generality we assume that $\varepsilon_n \to \varepsilon$, $\varepsilon_n v + (1 - \varepsilon_n)I \to v_0 \in \mathscr{M}_+^o$, and then we immediately get a contradiction with our hypotheses. The theorem is proved.

Definition 2.13. Given elliptic equation (1.1), its elliptic branch (2.2) and an $\varepsilon > 0$, we call the following equation the *ε-elliptic regularization* of (1.1)

$$(u_{x^i x^j}(x) + \varepsilon\delta_{ij}\Delta u(x)) \in \partial\Theta, \quad x \in D, \tag{2.3}$$

which, as is easy to check, is an elliptic branch of the equation

$$F(u_{x^i x^j} + \varepsilon\delta_{ij}\Delta u) = 0.$$

For $t \in [0, 1]$ consider the following elliptic branches

$$(tu_{x^i x^j}(x) + (1 - t)\delta_{ij}\Delta u(x)) \in \partial\Theta, \quad x \in D, \tag{2.4}$$

of the equations

$$F(tu_{x^i x^j} + (1 - t)\delta_{ij}\Delta u) = 0.$$

We call the family of equations (2.4) *the right interval between equation* (1.1) *and the equation* $\Delta u = c$, where the constant c is defined as a unique constant (see Remark 2.9) such that $c(\delta_{ij}) \in \partial\Theta$.

A justification of these definitions can be found in Remark 4.3 below.

Remark 2.14. The ε-elliptic regularization (2.3) of (1.1) is ε_1-nondegenerate elliptic, where $\varepsilon_1 = \varepsilon(1 + \varepsilon\sqrt{d})^{-1}$. Indeed, define $\Theta_1 = \{w : w + \varepsilon I \operatorname{tr} w \in \Theta\}$ and take

$$w \in \partial\Theta_1, \quad \xi \in \mathbb{R}^d \setminus \{0\}, \quad ||v|| < \varepsilon_1|\xi|^2, \quad v_1 := \xi\xi^* + \varepsilon I|\xi|^2 + v + \varepsilon I \operatorname{tr} v.$$

Then for $w_1 = w + \xi\xi^* + v$ we have

$$w_1 + \varepsilon I \operatorname{tr} w_1 = w + \varepsilon I \operatorname{tr} w + v_1 \in \Theta, \qquad w_1 \in \Theta_1,$$

since $w + \varepsilon I \operatorname{tr} w \in \Theta$ and

$$v_1 > \xi\xi^* + |\xi|^2 I(\varepsilon - \varepsilon_1 - \varepsilon\varepsilon_1\sqrt{d}) = \xi\xi^* \geq 0.$$

This implies that $\operatorname{dist}(w + \xi\xi^*, \partial\Theta_1) \geq \varepsilon_1|\xi|^2$, as we have claimed.

It is worth noting that if equation (1.1) is linear (nontrivial) and elliptic in the ordinary sense, then it is elliptic in the sense of Definition 2.2 too. Also, a linear elliptic equation nondegenerate in the usual sense is nondegenerate in our sense as well. The following theorem generalizes this fact. It shows, actually, that general fully nonlinear elliptic equations studied in the literature are elliptic in our sense.

Theorem 2.15. (i) *If* $F(u_{ij} - \xi^i\xi^j) < F(u_{ij}) < F(u_{ij} + \xi^i\xi^j)$ *for any vector* $\xi \in \mathbb{R}^d \setminus \{0\}$ *and symmetric* $d \times d$ *matrix* (u_{ij}) *(and if* $\Gamma \neq \emptyset$*), then equation* (1.1) *coincides with its only elliptic branch.*

(ii) *If there exists a constant* $\delta > 0$ *such that for any vector* $\xi \in \mathbb{R}^d$ *and* $d \times d$ *matrix* (u_{ij}) *we have*

$$\delta|\xi|^2 \leq F(u_{ij} + \xi^i\xi^j) - F(u_{ij}) \leq \delta^{-1}|\xi|^2, \tag{2.5}$$

then equation (1.1) *coincides with its* $\delta^2/\sqrt{d}$*-nondegenerate elliptic branch.*

Moreover, the function F *is elliptically convex in the sense that for any* $d \times d$ *symmetric matrices* w_1, w_2 *there is a matrix* a *such that*

$$\delta|\xi|^2 \leq a^{ij}\xi^i\xi^j \leq \delta^{-1}|\xi|^2$$

for any $\xi \in \mathbb{R}^d$ *and*

$$F(w_1) - F(w_2) = a^{ij}(w_{1ij} - w_{2ij}).$$

Proof. (i) As in Remark 2.4 for any $w \in \mathbb{S}^d$ and $v \in \mathscr{M}_+^o$ we have $F(w - v) < F(w) < F(w + v)$. Moreover, $\mathscr{M}_+^o$ is an open set, which implies that $\Gamma = \partial\Theta$, where $\Theta := \{F > 0\}$. In particular, equations (1.1) and (2.2) coincide. It is easy to see that Θ is an elliptic and even quasi-nondegenerate set, so that equation (2.2) is an elliptic branch of equation (1.1). It remains to note that equation (2.2) has only one branch.

(ii) We start with the second assertion in (ii). Observe that condition (2.5) means (and means only) that for any $\xi \in \mathbb{R}^d$, $w \in \mathbb{S}^d$ there exist numbers $a_\pm \in [\delta, \delta^{-1}]$ such that

$$F(w \pm \xi\xi^*) - F(w) = \pm a_\pm|\xi|^2. \tag{2.6}$$

Bearing this in mind, take w_1, $w_2 \in \mathbb{S}^d$ and note that for λ_i, ξ_i defined as eigenvalues and unit eigenvectors of $w_1 - w_2$ we have

$$w_1 = w_2 + \sum_{i=1}^d \lambda_i\xi_i\xi_i^*.$$

Hence by (2.6) for some numbers $a_i \in [\delta, \delta^{-1}]$ we get

$$F(w_1) = F(w_2) + \sum_{i=1}^d a_i\lambda_i = F(w_2) + \sum_{i=1}^d a_i \operatorname{tr}\xi_i\xi_i^*(w_1 - w_2),$$

and it remains to take $a = \sum a_i\xi_i\xi_i^*$.

Then we prove that the only elliptic branch of (1.1) is $\delta^2/\sqrt{d}$-nondegenerate. To this end we note that the above argument shows that F is Lipschitz continuous: for any w_1, $w_2 \in \mathbb{S}^d$

$$|F(w_1) - F(w_2)| \le \delta^{-1} \sum |\lambda_i| \le \delta^{-1}\sqrt{d}\, ||w_1 - w_2||.$$

Next, if $w, v \in \Gamma$, $\xi \in \mathbb{R}^d$, and $||v - (w + \xi\xi^*)|| = \operatorname{dist}(w + \xi\xi^*, \Gamma)$, then

$$\begin{aligned}\delta|\xi|^2 = \delta|\xi|^2 + F(w) &\le F(w + \xi\xi^*) = F(w + \xi\xi^*) - F(v)\\ &\le \delta^{-1}\sqrt{d}\, ||v - (w + \xi\xi^*)|| = \delta^{-1}\sqrt{d}\ \operatorname{dist}(w + \xi\xi^*, \Gamma).\end{aligned}$$

The theorem is proved.

Now let us consider several simple examples of nonlinear equations.

Example 2.16. For $d = 1$ consider the simplest linear equation $0 \cdot u'' = 0$. Here $\Gamma = R$ and we can take as Θ *any* set like (c, ∞), where c is an arbitrary constant. We see that our equation is *uniformly elliptic* and has infinitely many elliptic branches given by the equations $u'' = c$.

Example 2.17. Take $d = 1$ and the equation $|u''| = 1$ on $(0, 1)$. Here $\Gamma = \{1\} \cup \{-1\}$, and it is easy to check that we can take as Θ any of intervals $(1, \infty)$, $(-1, \infty)$. These two sets define two different elliptic branches: $u'' = 1$ and $u'' = -1$. These equations are linear nondegenerate elliptic equations.

Example 2.18. For $d = 2$ consider the equation $u_{x^1x^1}u_{x^2x^2} = 0$. It is easy to see that this equation has four different elliptic branches corresponding to the domains $\{u_{11} > 0, u_{22} > 0\}$, $\{u_{11} > 0 \text{ or } u_{22} > 0\}$, $\{u_{11} > 0\}$, and $\{u_{22} > 0\}$. These branches can be written as the following equations

$$\min(u_{x^1x^1}, u_{x^2x^2}) = 0, \quad \max(u_{x^1x^1}, u_{x^2x^2}) = 0, \quad u_{x^1x^1} = 0, \quad u_{x^2x^2} = 0.$$

All the branches are degenerate elliptic equations.

Example 2.19. Take equation (1.2). In the 3-space of 2×2 symmetric matrices the surface $\Gamma = \{2u_{11}^2 + 5u_{11}u_{22} + 2u_{22}^2 = 1\}$ is a cylinder in the u_{12} direction, whose intersection with the (u_{11}, u_{22})-plane is a hyperbola with the asymptotes $2u_{11} + u_{22} = 0$ and $u_{11} + 2u_{22} = 0$. Furthermore, one branch, say, Γ_1, of this hyperbola intersects the positive quadrant and another one, Γ_2, intersects the negative quadrant. Here we have three connected open components of $\mathbb{R}^3 \setminus \Gamma$. One of them whose intersection with (u_{11}, u_{22})-plane lies "above" Γ_1 defines the positive elliptic set, and the component opposite to it defines the negative elliptic one. Equation (1.2) is elliptic having two branches:

$$(u_{x^1x^1}, u_{x^2x^2}) \in \Gamma_1 \quad \text{and} \quad (u_{x^1x^1}, u_{x^2x^2}) \in \Gamma_2.$$

It is easy to see that we can rewrite these branches differently:

$$2u_{x^1x^1}^2 + 5u_{x^1x^1}u_{x^2x^2} + 2u_{x^2x^2}^2 = 1,$$

$$\Delta u \ge c, \tag{2.7}$$

for the first branch, where c is any number in $(-2/3, 2/3]$; and

$$2u_{x^1x^1}^2 + 5u_{x^1x^1}u_{x^2x^2} + 2u_{x^2x^2}^2 = 1,$$

$$\Delta u < 0 \tag{2.8}$$

for the second one. As is easy to understand, both branches are uniformly elliptic.

Remark 2.20. The set of all functions satisfying (2.7) is one of the right sets (of type $\mathscr{C}$ from Section 1) where we should look for solutions of (1.2) and where we can even find them (see Theorem 3.4 below). The only real role of inequality (2.7) is that it defines a branch of equation (1.2). Inequality (2.8) defines another "good" class of functions.

Example 2.21. Now take equation (1.3). As in the previous example, here the set Γ is also a cylinder with the base having the same asymptotes, but this time the base lies in negative quadrants where $u_{11}u_{22} < 0$. It is easy to check that there is no combination of the three connected open components satisfying the requirements of Definition 2.1. Therefore, this equation *we do not call elliptic.*

Observe that, actually, there is a class of functions $\mathscr{C}$ such that for any $u \in \mathscr{C}$ at any point in D the matrix $\partial F/\partial u_{ij}(u_{x^ix^j}(x))$ is definite under the condition that u satisfies (1.3). Indeed, it turns out that the largest such set is $\mathscr{C} = \{u : |\Delta u| \geq 1/\sqrt{18}\}$. Moreover, for some boundary data there even exists a solution of the equation in $\mathscr{C}$. Nevertheless, the behavior of this equation is quite unnatural for equations which we would like to call elliptic. For example, if $D = \{|x| < 1\}$ and we take any smooth boundary data which is sufficiently close to a constant, then in $\mathscr{C}$ there is no smooth solution of the equation with this data. The reason for this is that either $\Delta u \geq 1/\sqrt{18}$ in D or $\Delta u \leq -1/\sqrt{18}$ in D, and if, for instance, $\Delta u \geq 1/\sqrt{18}$, then somewhere in D the function u will attain its minimum value. At this point $u_{x^1x^1}u_{x^2x^2} \geq 0$, and at this point u cannot satisfy the equation. It is useful to note that all this happens in spite of the fact that inequality $\Delta u \geq 1/\sqrt{18}$ looks very much like (2.7).

Example 2.22. The equation $u_{x^1x^1}u_{x^2x^2} = 1$ is an elliptic one in a domain $D \subset \mathbb{R}^2$. It has two branches, which are quasi-nondegenerate but not uniformly elliptic equations. Adding Δu to the left-hand side of this equation does not make it uniformly elliptic. Indeed, the equation $u_{x^1x^1}u_{x^2x^2} + \Delta u = 1$ (equivalent to $(u_{x^1x^1} + 1)(u_{x^2x^2} + 1) = 2$) is not a uniformly elliptic equation. To make a uniformly elliptic perturbation of the equation we should follow Definition 2.13, and then we see, for instance, that system of relations (2.7) is, actually, a 1-elliptic regularization of the equation $u_{x^1x^1}u_{x^2x^2} = 1$.

In connection with this example it is useful to note that the equation

$$u_{x^1x^1}u_{x^2x^2}u_{x^3x^3} = 1$$

in a domain $D \subset \mathbb{R}^3$ is also quasi-nondegenerate elliptic and *with only one elliptic branch* but that the equation $u_{x^1x^1}u_{x^2x^2}u_{x^3x^3} + \Delta u = 1$ is not elliptic at all. The reason for the latter is that the section of the corresponding set Γ by the plane $u_{33} = 1$ is defined by the equation $(2u_{11} + 1)(2u_{22} + 1) = -1$ which has no elliptic branches.

Example 2.23. We will see later (cf. Remark 6.7) that the Monge-Ampère equation $\det(u_{x^ix^j}) = 1$ is an elliptic quasi-nondegenerate equation and, moreover,

that the system

$$\det(u_{x^ix^j}) = 1, \qquad (u_{x^ix^j}) > 0,$$

describes its elliptic branch corresponding to the elliptic set $\Theta = \{w : w > 0, \det w > 1\}$. This means that we can consider the equation in the set of all convex functions. According to Definition 2.13 the family of systems of relations

$$\det(tu_{x^ix^j} + (1-t)\delta_{ij}\Delta u) = 1, \qquad (tu_{x^ix^j} + (1-t)\delta_{ij}\Delta u) > 0,$$

is the right interval between the Monge-Ampère equation and the Laplace equation $\Delta u = 1$. The second relation in this system, actually, defines the right class of functions $\mathscr{C}_t$ about which we were talking in Section 1. By the way, a different method of joining our equation with a simpler (or "model") one is usually applied in the literature (see, for instance, [2], [3]). Namely, one takes any function ψ which is smooth and strictly convex in D and has the same boundary data as for the solution we are looking for, and one considers the family of equations

$$\det(u_{x^ix^j}) = t + (1-t)\det(\psi_{x^ix^j}).$$

It is not hard to see that all these equations are quasi-nondegenerate elliptic in our sense.

Definition 2.24. Given an elliptic equation (1.1) (or (2.1)), we say that a function u is an *admissible solution* in D if u is a solution in D of any elliptic branch of the equation (the branch should be the same in the whole of D).

Note, for instance, that $u(x, y) = x^2 - y^2$ is not an admissible solution of the elliptic equation $u_{xx}^2 u_{yy}^2 = 16$.

Remark 2.25. This definition is sufficient for the purposes of the present article. However, in applications the following more general definition might make sense too.

Take a continuous function $F(u_{ij}, u_i, u, x)$ defined on $\mathbb{S}^d \times \mathbb{R}^d \times R \times \mathbb{R}^d$ such that the set $\Gamma(x) = \{(u_{ij}, u_i, u) : F(u_{ij}, u_i, u, x) = 0\}$ is nonempty for any $x \in D$.

We say that the equation

$$F(u_{x^ix^j}(x), u_{x^i}(x), u(x), x) = 0, \qquad x \in D, \tag{2.9}$$

is *elliptic* if for any $x \in D$ there exists a nonempty open connected set $\Theta(x) \subset \mathbb{S}^d \times \mathbb{R}^d \times R$, $\Theta(x) \neq \mathbb{S}^d \times \mathbb{R}^d \times R$, such that

(i) $\partial\Theta(x) \subset \Gamma(x)$, $\Theta(x) = \bar{\Theta}(x) \setminus \partial(\bar{\Theta}(x))$,

(ii) if $(u_{ij}, u_i, u) \in \partial\Theta(x)$, $\xi \in \mathbb{R}^d$, then $(u_{ij} + \xi^i\xi^j, u_i, u) \in \bar{\Theta}(x)$,

(iii) the set $\Theta(x)$ depends continuously on $x \in D$.

If such a set Θ does exist we call the equation

$$(u_{x^ix^j}(x), u_{x^i}(x), u(x)) \in \partial\Theta(x), \quad x \in D, \tag{2.10}$$

an *elliptic branch of* (2.9) defined by $\Theta(x)$.

We say that a function $u \in C^2(D)$ is an *admissible solution* of (2.9) if it is a solution of one of its elliptic branches.

We say that a function $u \in C(D)$ is an *admissible viscosity solution* of (2.9) given an elliptic branch defined by $\Theta(x)$ if for any function $\phi \in C^2(D)$

(a) at any point $x_0 \in D$ where $u - \phi$ attains its local minimum equal to zero, we have $(\phi_{x^i x^j}(x_0), \phi_{x^i}(x_0), \phi(x_0)) \notin \Theta(x_0)$,

(b) at any point $x_0 \in D$ where $u - \phi$ attains its local maximum equal to zero, we have $(\phi_{x^i x^j}(x_0), \phi_{x^i}(x_0), \phi(x_0)) \in \bar{\Theta}(x_0)$.

It is easy to check that from the point of view of these definitions the functions $1 - |x|$, $|x| - 1$ are both admissible viscosity solutions of the equation $|u'| - 1 = 0$ on $(-1, 1)$ but the first one solves its branch defined by $\Theta = \{|u'| < 1, x \in (-1, 1)\}$ and the second one solves the branch corresponding to $\Theta = \{|u'| > 1, x \in (-1, 1)\}$. In our sense the equations $|u'| - 1 = 0$ and $-|u'| + 1 = 0$ are *equivalent* (which is rather reasonable) in contrast to the situation with the usual definition of viscosity solutions (cf., for instance, [7, Remark 2.5]).

Note also that, for instance, $u(x) = x^2 \operatorname{sign} x$ is not an admissible viscosity solution of the elliptic equation $|u''| = 2$ since $u''(x) \in \partial\Theta(x)$ for $x \neq 0$, where $\Theta(x) = (-2, \infty)$ for $x < 0$ and $\Theta(x) = (2, \infty)$ for $x > 0$ so that $\Theta(x)$ is discontinuous in x.

3. Canonical form of elliptic branches

After the reader has realized that his equation is an elliptic one and understands in which branch of the equation he is interested, he might like to read corresponding books or articles. Then he sees that in the literature we are not dealing with equations of the form (2.2) but rather of the usual form like (1.1) or (2.9). The known results and methods of proving a priori estimates and of proving existence theorems on the basis of these estimates are all adapted to equations in the usual form. Here we want to discuss how to convert elliptic branches like (2.2) into the usual form.

Definition 3.1. Given an elliptic equation (1.1) (or (2.1)) and its elliptic branch (2.2) defined by an elliptic set Θ, we call equation (1.4) a *canonical form* of branch (2.2) if the function $\tilde{F}$ has the following properties:

(1) $\tilde{F}(u_{ij}) > 0$ in Θ.
(2) $\tilde{F}(u_{ij}) < 0$ outside $\bar{\Theta}$.
(3) $\tilde{F}(u_{ij}) = 0$ on $\partial\Theta$.

Obviously, there are infinitely many canonical forms for any given elliptic branch of an elliptic equation. The following theorem brings to the end the investigation of the possibility of applying the general theory of fully nonlinear elliptic equations to a given equation like (1.1). This theorem should be compared with Theorem 2.15.

Theorem 3.2. *Let Θ be an elliptic set and equation* (2.2) *be elliptic* (*for instance, be an elliptic branch of* (1.1)). *Define*

$$\tilde{F}(u_{ij}) = \operatorname{dist}((u_{ij}), \partial\Theta) \quad \textit{for } (u_{ij}) \in \bar{\Theta},$$
$$\tilde{F}(u_{ij}) = -\operatorname{dist}((u_{ij}), \partial\Theta) \quad \textit{for } (u_{ij}) \in \mathbb{S}^d \setminus \Theta.$$

Then

$$w \in \partial\Theta \iff \tilde{F}(w) = 0, \tag{3.1}$$

and in particular, equation (1.4) *is equivalent to equation* (2.2). *Furthermore, for any* $\xi \in \mathbb{R}^d$, $(u_{ij}) \in \mathbb{S}^d$

$$0 \le \tilde{F}(u_{ij} + \xi^i\xi^j) - \tilde{F}(u_{ij}) \le |\xi|^2. \tag{3.2}$$

Moreover, the function $\tilde{F}$ *is elliptically convex in the sense that for any* (u_{ij}), $(v_{ij}) \in \mathbb{S}^d$ *the difference* $\tilde{F}(u_{ij}) - \tilde{F}(v_{ij})$ *can be represented as* $a^{ij}(u_{ij} - v_{ij})$ *with a nonnegative symmetric matrix* a. *Finally, if equation* (2.2) *is* δ*-nondegenerate, then*

$$\delta|\xi|^2 \le \tilde{F}(u_{ij} + \xi^i\xi^j) - \tilde{F}(u_{ij}). \tag{3.3}$$

Proof. Assertion (3.1) is obvious, and the second inequality in (3.2) follows easily from the fact that the distance from a set is a Lipschitz continuous function with the Lipschitz constant 1. To prove the first one for $u = (u_{ij}) \in \bar{\Theta}$ it suffices to introduce $w \in \partial\Theta$ such that $||(u + \xi\xi^*) - w|| = \tilde{F}(u + \xi\xi^*)$ and to note that since $w - \xi\xi^* \notin \Theta$ (see Remark 2.6), we have $\tilde{F}(u) \le ||u - (w - \xi\xi^*)||$. Practically the same argument proves inequality (3.3) for $u \in \bar{\Theta}$ if equation (2.2) is δ-nondegenerate.

If $u + \xi\xi^* \in \mathbb{S}^d \setminus \bar{\Theta}$, then $u \in \mathbb{S}^d \setminus \bar{\Theta}$, and for $w \in \partial\Theta$ such that $||u - w|| = -\tilde{F}(u)$ we have $w + \xi\xi^* \in \bar{\Theta}$ and $||u - w|| = ||(u + \xi\xi^*) - (w + \xi\xi^*)|| \ge -\tilde{F}(u + \xi\xi^*)$. In the same way we get (3.3) for $u + \xi\xi^* \in \mathbb{S}^d \setminus \bar{\Theta}$ if equation (2.2) is δ-nondegenerate. But if $(u_{ij}) \in \mathbb{S}^d \setminus \bar{\Theta}$ and $(u_{ij} + \xi^i\xi^j) \in \bar{\Theta}$, these inequalities are almost obvious (consider the straight segment $[(u_{ij}), (u_{ij} + \xi^i\xi^j)]$).

The remaining assertion about the difference $\tilde{F}(u_{ij}) - \tilde{F}(v_{ij})$ can be obtained exactly as in the proof of Theorem 2.15. The theorem is proved.

An immediate consequence of this theorem and of results from [7] is the following

Theorem 3.3. *Let* D *be a bounded smooth domain and* ϕ *be a continuous function on* ∂D. *Assume that equation* (1.1) *has a uniformly elliptic branch. Then this equation with the boundary condition* $u = \phi$ *on* ∂D *has an admissible viscosity solution* $u \in C(\bar{D})$ (*see Remark* 2.25). *Moreover, every uniformly elliptic branch of* (1.1) *has its own unique admissible viscosity solution* $u \in C(\bar{D})$.

Note that in Theorem 3.2 the function $\tilde{F}$ is obviously concave if Θ is convex and is convex if the complement of Θ is convex. If we combine this with results from [17], then we obtain

Theorem 3.4. *Let* D *be a bounded domain of class* $C^{2+\alpha}$ *where* $\alpha \in (0, 1)$, *and let* $\phi \in C^{2+\alpha}(\mathbb{R}^d)$. *Assume that equation* (1.1) *has a uniformly elliptic branch defined by a domain* Θ *such that either* Θ *or its complement is convex. Then this equation with the boundary condition* $u = \phi$ *on* ∂D *has an admissible solution* $u \in C^{2+\beta}(\bar{D})$, *where* $\beta \in (0, 1)$. *Moreover, the elliptic branch* (2.2) *with the given boundary condition has its own unique admissible solution* $u \in C^{2+\beta}(\bar{D})$.

This theorem applies to Examples 2.17 and 2.19.

4. The case of convex domains Θ and Bellman equations

Theorem 3.4 is one in a whole series of results related to a very important and so far the only case of fully nonlinear elliptic equations for which a "good" theory of existence of *smooth* solutions is developed. In this theory we need not only conditions like (3.2) and (3.3) about the function F from (1.1) but also *convexity or concavity* of F with respect to (u_{ij}). Therefore, after coming from equation (1.1) to equation (1.4) with $\tilde{F}$ defined in Theorem 3.2, we want the function $\tilde{F}$ to be concave or convex. This reduces to the situation when either Θ or its complement is convex. In what follows we consider only the case when Θ is convex. The reader can easily reformulate all our results for the other case if he replaces the unknown function u by $-u$ and notices that by Remark 2.4 the set $-\mathbb{S}^d \setminus \bar{\Theta}$ is an elliptic set if the set Θ is elliptic.

In the case of convex Θ we can find a different formula for the function $\tilde{F}$ from Theorem 3.2.

Recall that given a convex open set $B \subset \mathbb{S}^d$ and $a \in \mathbb{S}^d$, $f \in R$, we say that the plane in $\mathbb{S}^d$ defined by the equation $\operatorname{tr} aw + f = 0$ is a supporting plane of B if $\inf_B \operatorname{tr} aw = -f$.

Theorem 4.1. *Let equation* (2.2) *be an elliptic branch of* (1.1) *defined by a convex domain* Θ. *Define* $\mathscr{A}$ *as the set of all* $a \in \mathscr{M}_+$ *with* $\operatorname{tr}(a^2) = 1$ *such that for a number* f *the plane in* $\mathbb{S}^d$ *described by the equation* $\operatorname{tr} aw + f = 0$ *is a supporting plane of* Θ. *Such an* f *is defined uniquely, and if we denote it as* $f(a)$, *then*

$$f(a) = -\inf_{w \in \Theta} \operatorname{tr} aw,$$

and the function $\tilde{F}$ *from Theorem* 3.2 *admits the representation*

$$\tilde{F}(u_{ij}) = \inf_{a \in \mathscr{A}} (a^{ij} u_{ij} + f(a)). \tag{4.1}$$

Proof. For any $a \in \mathscr{A}$, $u_1 \in \bar{\Theta}$, $u_2 \in \mathbb{S}^d \setminus \bar{\Theta}$, and w_i defined as the point of the plane $a^{ij} w_{ij} + f(a) = 0$ closest to u_i we have

$$\operatorname{dist}(u_1, \partial\Theta) \le \|u_1 - w_1\| = \operatorname{tr} a(u_1 - w_1) = \operatorname{tr} au_1 + f(a),$$
$$\operatorname{dist}(u_2, \partial\Theta) \ge \operatorname{tr} a(w_2 - u_2) = -\operatorname{tr} au_2 - f(a).$$

It follows that the right-hand side of (4.1) is not less than its left-hand side.

To show the opposite inequality recall that, as is well known, the graph of the function $\tilde{F}$ coincides with the concave hull of the family of graphs of linear functions $z = a^{ij} u_{ij} + f$ on $\mathbb{S}^d$ touching $\bar{\Theta} \times \{0\}$ and such that each function $a^{ij} u_{ij} + f$ has a unit gradient (in variables u_{ij}) looking "inside" Θ for $(u_{ij}) \in \partial\Theta$. Therefore, we must only show that given any such linear function, we have $a = (a^{ij}) \in \mathscr{A}$.

The fact that $\operatorname{tr}(a^2) = 1$ is true since the length of the gradient of $a^{ij} u_{ij} + f(a)$ equals one. Moreover, if $(u_{ij}) \in \partial\Theta$ and $a^{ij} u_{ij} + f(a) = 0$, then for any $(v_{ij}) \in \mathbb{S}^d$ we have $a^{ij}(u_{ij} + v_{ij}) + f(a) > 0$ whenever $(u_{ij} + v_{ij}) \in \Theta$. Since $(u_{ij}) + \mathscr{M}_+^o \in \Theta$, it follows that $a^{ij}(u_{ij} + \xi^i \xi^j) + f(a) \ge 0$ and

$$a^{ij} \xi^i \xi^j = a^{ij}(u_{ij} + \xi^i \xi^j) + f - (a^{ij} u_{ij} + f) \ge 0.$$

for any $\xi \in \mathbb{R}^d$. Thus the theorem is proved.

Remark 4.2. Theorem 4.1 allows us to express the function $\tilde{F}$ *explicitly in terms of the initial function* F (and the domain Θ once Θ is convex) in the following way: If F is continuously differentiable in a neighborhood of $\partial\Theta$, $F \geq 0$ in Θ, and $(F_{u_{ij}}) \neq 0$ on $\partial\Theta$, then

$$\tilde{F}(u_{ij}) = \inf_{w \in \partial\Theta} \|(F_{u_{ij}}(w))\|^{-1}\{F_{u_{ij}}(w)u_{ij} - F_{u_{ij}}(w)w_{ij}\}.$$

Remark 4.3. Observe that for $a \in \mathscr{M}_+$ with $\operatorname{tr} a^2 = 1$ we have $1 \leq \operatorname{tr} a \leq \sqrt{d}$. Therefore, equation (1.4), which is equivalent to (2.2) by virtue of Theorem 3.2, by Theorem 4.1 is also equivalent to

$$\hat{F}(u_{x^i x^j}) = 0, \tag{4.2}$$

where

$$\hat{F}(u_{ij}) := \inf_{\omega \in \Omega}(\omega_{ij}u_{ij} + g(\omega)),$$

$$\begin{gathered} \Omega := \{(\operatorname{tr} a)^{-1}a : a \in \mathscr{A}\}, \\ g((\operatorname{tr} a)^{-1}a) = (\operatorname{tr} a)^{-1}f(a) = -\inf_{w \in \Theta}(\operatorname{tr} a)^{-1} \operatorname{tr} aw. \end{gathered} \tag{4.3}$$

Moreover,

$$(u_{ij}) \in \partial\Theta \iff \hat{F}(u_{ij}) = 0. \tag{4.4}$$

An advantage of equation (4.2) is that for $\omega \in \Omega$ we have $\operatorname{tr} \omega = 1$, and in terms of the function $\hat{F}$ the ε-elliptic regularization (2.3) of equation (1.1) (see Definition 2.13) looks quite natural:

$$\hat{F}(u_{x^i x^j}) + \varepsilon\Delta u = 0.$$

Moreover, the right interval between equation (1.1) and the Laplace equation $\Delta u = c$, as introduced in the same definition, now looks natural as well:

$$\hat{F}(tu_{x^i x^j}) + (1 - t)\Delta u = 0.$$

This is the main reason why we introduced Definition 2.13.

Equations with functions $\tilde{F}$, defined as in (4.1) as an upper (or lower) bound of a set of linear functions, first arose in the theory of controlled diffusion processes, where they are called Bellman equations.

Definition 4.4. Let Ω be a set, and for every $\omega \in \Omega$ let a matrix $a(\omega) \in \mathscr{M}_+$ and a function $f(\omega, x)$ on D be defined. Define

$$H(u_{ij}, x) = \inf_{\omega \in \Omega}(a^{ij}(\omega)u_{ij} + f(\omega, x)). \tag{4.5}$$

Then the equation

$$H(u_{x^i x^j}(x), x) = 0, \qquad x \in D, \tag{4.6}$$

is called a *Bellman equation with constant coefficients* (and without lower order terms). If for any unit $\xi \in \mathbb{R}^d$ we have

$$\sup_{\omega \in \Omega} a^{ij}(\omega)\xi^i\xi^j > 0,$$

the equation is called *weakly nondegenerate*, and if for any constant $N > 0$ there is a $\delta > 0$ such that

$$\frac{\partial^+}{\partial t} H(u_{ij} + t\xi^i\xi^j, x)|_{t=0} \geq \delta$$

whenever $||(u_{ij})|| \leq N$, $H(u_{ij}, x) = 0$, $x \in D$, $|\xi| = 1$, then the equation is called *quasi-nondegenerate.*

Remark 4.5. After converting equation (2.2) into the Bellman equation

$$\inf_{a \in \mathscr{A}} (a^{ij} u_{x^i x^j}(x) + f(a)) = 0, \qquad x \in D, \tag{4.7}$$

the reader can look, for example, in the book [17] for further information about the solvability of such equations. Then he will find that it is usually assumed there that $f(a)$ is bounded on $\mathscr{A}$. However, there are many cases when for $\mathscr{A}$ taken from Theorem 4.1 the function $f(a)$ is unbounded. In connection with this, it is worth mentioning Lemma 6.3.6 of [15] which says that in the situation of Theorem 4.1 equation (4.7) *is equivalent* to the equation

$$\inf_{a \in \mathscr{A}} (1 + |f(a)|)^{-1} (a^{ij} u_{x^i x^j}(x) + f(a)) = 0, \qquad x \in D,$$

which is called *normalized* Bellman equation.

Remark 4.6. If the elliptic branch (2.2) (with convex Θ) of equation (1.1) is quasi-nondegenerate in the sense of Definition 2.1, then considered as a Bellman equation this branch is quasi nondegenerate in the sense of Definition 4.4 too.

To show this we note that for w satisfying $\tilde{F}(w) = 0$ and a unit ξ, the function $\tilde{F}(w + t\xi\xi^*)$ is concave, equals zero for $t = 0$, and is strictly bigger than zero for $t > 0$ ($w + t\xi\xi^* \in \Theta$). Therefore, its right derivative at zero is strictly positive and remains only to use that this derivative is lower semicontinuous in w, ξ.

Remark 4.7. The same argument involving the semicontinuity shows that if $H(u_{ij}, x)$ is a continuous function on $\mathbb{S}^d \times \bar{D}$ and if

$$\frac{\partial^+}{\partial t} H(u_{ij} + t\xi^i\xi^j, x)|_{t=0} > 0$$

whenever $H(u_{ij}, x) = 0$, $x \in \bar{D}$, and $|\xi| = 1$, then equation (4.6) is quasi-nondegenerate.

Remark 4.8. The weak nondegeneracy of (4.6) is equivalent to the fact that the function H actually depends on all second-order derivatives of u, that is, there is no proper subspace of $\mathbb{R}^d$ such that $H(u_{x^i x^j}, x)$ can be expressed in terms of the second-order derivatives of u along this subspace only (see [17, Section 2.2]).

Remark 4.9. If equation (4.6) is quasi-nondegenerate and $a(\omega)$ is bounded, then for any N we have

$$\inf_{\omega\in\Omega(\delta)}(a^{ij}(\omega)u_{ij}+f(\omega,x))=0,$$

whenever $\|(u_{ij})\|\le N$, $x\in D$, $H(u_{ij},x)=0$, where for the δ taken from Definition (4.3)

$$\Omega(\delta)=\{\omega\in\Omega: a^{ij}(\omega)\xi^i\xi^j\ge \tfrac{1}{2}\delta|\xi|^2 \quad \forall\xi\in\mathbb{R}^d\}.$$

Indeed, fix u_{ij}, x and take ω_n such that

$$a^{ij}(\omega_n)u_{ij}+f(\omega_n,x)\to H(u_{ij},x)=0. \tag{4.8}$$

Then for any unit ξ, for the following (concave) functions of t

$$h_n(t):=a^{ij}(\omega_n)(u_{ij}+t\xi^i\xi^j)+f(\omega_n,x),\quad h(t):=H(u_{ij}+t\xi^i\xi^j,x)$$

we have $h_n\ge h$ and $h_n(0)\to h(0)$. It follows easily that

$$\liminf_{n\to\infty}\frac{\partial}{\partial t}h_n(0)\ge\frac{\partial^+}{\partial t}h(0)\ge\delta,$$

which implies that for all large n (perhaps depending on ξ) we have

$$a^{ij}(\omega_n)\xi^i\xi^j\ge 3\delta/4.$$

Taking a finite number of vectors ξ dense enough on the unit sphere we will see that the same inequality with $\delta/2$ instead of $3\delta/4$ holds for all unit ξ, and this along with (4.8) implies that the value of the left-hand side in (4.5) will not change (at our particular u_{ij}, x) if we replace Ω by $\Omega(\delta)$.

Remark 4.10. Utilizing the fact that the difference of minimums is less than the maximum of differences, one sees that if there is at least one couple (u_{ij},x) such that $H(u_{ij},x)=0$ and the Bellman equation is quasi-nondegenerate, then it is weakly nondegenerate too.

Sometimes the set Ω and the function g from (4.3) can be characterized differently. In many cases the following theorem gives, in particular, the possibility to prove that a given equation is a weakly nondegenerate Bellman equation with constant and bounded coefficients.

Theorem 4.11. *Let Θ be an open convex set, and let equation* (2.2) *be elliptic. Let C be an open cone in $\mathbb{S}^d$ with vertex at the origin, and let t_0 be a number. Assume that $t_0I+\Theta\subset C$ and that for any $w\in C$ we have $tw\in\Theta$ for all t large enough. Then for the objects introduced in* (4.3) *we have*

$$\Omega=\{\omega\in\mathscr{M}_+:\operatorname{tr}\omega=1,\ \operatorname{tr}\omega w\ge 0\ \forall w\in C\}, \tag{4.9}$$

$$g(\omega)=-\inf_{w\in\Theta}\omega_{ij}w_{ij}. \tag{4.10}$$

In particular,

$$w\in\partial\Theta \iff \inf_{\omega\in\Omega}(\omega_{ij}w_{ij}+g(\omega))=0.$$

Moreover, $-t_1 \le g(\omega) \le t_0$, *where* t_1 *is the least* t *such that* $tI \in \bar{\Theta}$. *Finally, if* $\operatorname{tr} w \ge 0$ *for all* $w \in C$, *then the Bellman equation*

$$\inf_{\omega \in \Omega} (\omega_{ij} u_{x^i x^j} + g(\omega)) = 0$$

is weakly nondegenerate.

Proof. Denote by Ω_1 the right-hand side in (4.9). If $\omega \in \Omega$, then there is an $a \in \mathscr{A}$ such that $\omega \operatorname{tr} a = a$, $g(\omega) \operatorname{tr} a = -\inf_\Theta a^{ij} w_{ij}$. In particular, $\operatorname{tr} \omega = 1$. Furthermore, for any $w \in C$ and all large t we have $tw \in \Theta$, and $-g(\omega) \operatorname{tr} a \le t a^{ij} w_{ij}$. It follows that $a^{ij} w_{ij} \ge 0$, $\omega \in \Omega_1$ and $\Omega \subset \Omega_1$.

To prove the opposite inclusion take $\omega \in \Omega_1$ and define $a = \omega(\operatorname{tr} \omega^2)^{-1/2}$. Then $\omega_{ij}(t_0 \delta_{ij} + w_{ij}) \ge 0$ for $w \in \Theta$. This implies that the function $a^{ij} w_{ij}$ is bounded from below on Θ, and thus $a \in \mathscr{A}$, and $\omega = a(\operatorname{tr} a)^{-1} \in \Omega$. We have proved (4.9).

Formula (4.10) follows immediately from (4.9) and (4.3). Our estimates of g are consequences of the inequalities

$$t_1 = \omega_{ij} t_1 \delta_{ij} \ge \inf_{w \in \Theta} \omega_{ij} w_{ij} \ge \inf_{w \in C} \omega_{ij}(w_{ij} - t_0 \delta_{ij}) \ge -t_0.$$

The last assertion of the theorem is obvious since $I/d \in \Omega$. The theorem is proved.

If we combine this theorem with Theorem 5.3 and Remark 5.4 (below), then we immediately get

Corollary 4.12. *If the conditions of Theorem* 4.11 *are satisfied, if* $\operatorname{tr} w \ge 0$ *for any* $w \in C$, *and if* D *is a strictly convex domain of class* C^4, *then for any* $\phi \in C^4(\mathbb{R}^d)$ *there is a unique function* $u \in C(\bar{D}) \cap C^{1,1}(\bar{D})$ *such that* $u = \phi$ *on* ∂D *and* $(u_{x^i x^j}) \in \partial\Theta$ *(a.e.) in* D. *If, in addition, equation* (2.2) *is quasi-nondegenerate, then* $u \in C^{2+\alpha}(\bar{D})$ *for an* $\alpha \in (0, 1)$.

The first part of this corollary for the case when D is a ball has been known for a long time (see [16]).

Remark 4.13. This corollary obviously applies to Example 2.22, to the first equation in Example 2.18, and to the second equation if we pass to negative branches or simply make a change of the unknown function $u \to -u$. In these last examples the requirement of convexity of D can be considerably relaxed if one uses the general theorem from [20] or Theorem 5.3 below. Then it turns out that we can take any C^4 bounded domain D which is strictly convex in a neighborhood of every one of its boundary point where the tangent line is either horizontal or vertical.

Remark 4.14. In Theorem 4.11 we can obviously replace the unit matrix I by any symmetric strictly positive matrix v_0; this is simply equivalent to a change of independent variables in the Bellman equation. This change allows us to replace the requirement $\operatorname{tr} w \ge 0$ by $\operatorname{tr} v_0 w \ge 0$ in the last assertion of the theorem. Then we see that our Bellman equation is weekly nondegenerate if (and only if) for a $v_0 \in \mathscr{M}_+^o$ the cone C lies in the "positive" half space of the Euclidean space $\mathbb{S}^d$ divided into two parts by the plane orthogonal to v_0 and passing through zero.

For the third and fourth equations in Example 2.18 we cannot find such a plane and these equations are not weakly nondegenerate.

To end this section we give a proof of Theorem 1.1 in the particular case of strictly convex domains D and constant coefficients l_k.

Theorem 4.15. *Under the hypotheses of Theorem* 1.1 *assume that the domain* D *is strictly convex and the functions* l_k *are constant. Then the assertions of this theorem hold true.*

We need the following lemma whose proof is deferred until Section 6 (see there the note preceding Lemma 6.6).

Lemma 4.16. *Define* C_m *as the open connected component containing* I *of the set* $\{w \in \mathbb{S}^d : P_m(w) > 0\}$. *Then*

(i) C_m *is an open convex cone in* $\mathbb{S}^d$ *with vertex at the origin containing* $\mathscr{M}_+^o$.

(ii) *For* $k = 0, \dots, m-1$ *we have* $P_k(w) > 0$ *on* C_m, *and for any* $v \in \partial C_m$ *we have* $(P_k/P_m)(w) \to \infty$ *as* $w \to v$, $w \in C_m$.

(iii) *For* $k = 1, \dots, m-1$ *the functions* $(P_k/P_m)(w)$ *are convex in* C_m.

Proof of Theorem 4.15. We will apply Lemma 4.16 to check the conditions of Corollary 4.12. Denote $c_k = (l_k^+)^{m-k+1}$ and define

$$\Theta = \left\{ w \in C_m : P_m(w) > \sum_{k=0}^{m-1} c_k P_k(w) \right\} = \left\{ w \in C_m : 1 > \sum_{k=0}^{m-1} c_k \frac{P_k}{P_m}(w) \right\}.$$

Note that by the last assertion of the lemma the open set Θ is convex, and by the second one $P_1(w) = \operatorname{tr} w > 0$ on C_m, so that in particular, $\Theta \neq \mathbb{S}^d$. Since for any $v \in \mathscr{M}_+^o$ the function $P_m(tv) = t^m P_m(v) > 0$ grows to infinity faster than $P_k(tv)$ whenever $k < m$, by Theorem 2.10 equation (2.2) is elliptic. By definition $\Theta \subset C_m$, and the corresponding assumption of Theorem 4.11, are satisfied with $t_0 = 0$. Furthermore, as we already mentioned $\operatorname{tr} w > 0$ on C_m. Thus, all conditions of Corollary 4.12 are satisfied, and to prove the first assertion of Theorem 1.1 it remains to prove that for

$$\Gamma = \left\{ w \in \mathbb{S}^d : P_m(w) = \sum_{k=0}^{m-1} c_k P_k(w) \right\}$$

we have

(4.11)

$$\partial\Theta = \left\{ w \in \Gamma : P_m(w + tI) > 0,\ P_m(w + tI) > \sum_{k=0}^{m-1} c_k P_k(w + tI)\ \forall t > 0 \right\}.$$

If $w \in \partial\Theta$, then, of course, $w \in \Gamma$ and by the ellipticity of (2.2) we have $w + tI \in \Theta$ for all $t > 0$. Therefore, the left set in (4.11) is a subset of the right one.

On the other hand, if $P_m(w + tI) > 0$ for all $t > 0$, then the function $P_m(sw + I)$ does not change sign for $s > 0$, and since $I \in C_m$, we have $sw + I \in C_m$, $w + tI \in C_m$ for $s, t > 0$. If in addition $P_m(w + tI) > \sum_{k=0}^{m-1} c_k P_k(w + tI)$ for all $t > 0$, then $w + tI \in \Theta$, and also, if else $w \in \Gamma$, then $w \in \partial\Theta$, which proves the opposite inclusion between the sides of (4.11).

To prove the second assertion of Theorem 1.1 we check that equation (2.2) is quasi-nondegenerate provided that $\max c_i > 0$. By definition we have to show that $w + t\xi\xi^* \in \Theta$ if $w \in \partial\Theta$, $|\xi| = 1$, and $t > 0$. Recall that in any case by ellipticity $w + t\xi\xi^* \in \bar{\Theta}$, and by convexity of Θ we have $w + t\xi\xi^* \in \Theta$ for all $t > 0$ whenever this inclusion holds at least for one $t > 0$.

Next note that when $\max c_i > 0$,

$$\bar{\Theta} \subset C_m. \tag{4.12}$$

Indeed, $P_m \geq 0$ on $\bar{\Theta}$, and if $P_m(w) = 0$ for a $w \in \bar{\Theta}$, then $w \in \partial C_m$, and for all $t > 0$ we have $w + tI \in \Theta$ and

$$1 > \sum_{k=0}^{m-1} c_k \frac{P_k}{P_m}(w + tI), \qquad (w + tI \in \Theta),$$

which contradicts Lemma 4.16(ii) for small t.

Now if $w \in \partial\Theta$, $|\xi| = 1$, and the number of t such that $w + t\xi\xi^* \in \partial\Theta$ is infinite, then the polynomials $P_m(w + t\xi\xi^*)$ and $\sum c_k P_k(w + t\xi\xi^*)$ as polynomials in t are identical. But we can find a $t_1 < 0$ such that $P_1(w + t_1\xi\xi^*) < 0$, which by Lemma 4.16(ii) implies that $w + t_1\xi\xi^* \notin \bar{C}_m$. On the other hand $w \in \bar{\Theta} \subset C_m$. Therefore, on $(t_1, 0)$ there is the largest t_0 such that $w + t_0\xi\xi^* \in \partial C_m$, and by the same lemma the above-mentioned identity cannot hold for t close to t_0 from the right. Thus the number of $t > 0$ for which $w + t\xi\xi^* \in \partial\Theta$ is finite (actually, empty as explained above), and we get the quasi-nondegeneracy of (2.2). The theorem is proved.

5. A THEOREM ABOUT BELLMAN EQUATIONS AND ITS APPLICATION

Corollary 4.12 relates to the case when the independent variable x does not enter equation (1.1); with other results from Sections 1 through 4 it shows what to expect from a given fully nonlinear equation at least if we "freeze coefficients" and disregard the dependence on u_x, u of the function defining the equation. If these results look satisfactory for a given equation the reader might like to investigate the equation further, and here we want to show how deeper investigations into properties of domains Θ allow us to apply the general theory to some equations with the independent variable x entering explicitly.

At first we state a general theorem about Bellman equations. Then we show how some properties of domains Θ defining elliptic branches are related to properties required in this theorem. As an example of applications we prove Theorem 1.1 in one more particular case. After this we translate other properties of domains Θ into properties of functions defining Bellman equations, and this enables us to prove Theorem 1.1 in its full generality.

Thus, our general manner of treating concrete fully nonlinear equations is to go from them to domains Θ, to try to get as much information of special kinds as possible about these domains, and then to refer to results from the general theory of fully nonlinear elliptic equations.

Let Ω be a set, integers $d_1, d_0 \geq 1$, D be a bounded domain in $\mathbb{R}^d$, a function $\psi \in C^4(\mathbb{R}^d)$, constants $K \in (1, \infty)$, $\delta \in (0, 1)$. Assume that for any $\omega \in \Omega$, $p \in \mathbb{R}^{d_0}$ we are given a $d \times d_1$ matrix $\sigma(\omega, p)$ and functions $f(\omega, p, x)$, $\phi(x)$ defined on $\mathbb{R}^d$. Denote by σ^k the kth column of σ and

$a = (1/2)\sigma\sigma^*$. By the way, in the preceding section we did not introduce the matrix σ, and it is worth noting that when we mentioned there the results of the present section we always meant $\sigma = \sqrt{2a}$.

Assumption 5.1. (a) $D = \{x \in \mathbb{R}^d : \psi(x) > 0\}$, $\|\psi\|_{C^4(\mathbb{R}^d)} \le K$.

(b) For any $\omega \in \Omega$ on ∂D

$$|\psi_x| \ge \delta, \quad a^{ij}(\omega, 0)\psi_{x^i x^j} \le -\delta.$$

(c) For any $\omega \in \Omega$, $p \in \mathbb{R}^{d_1}$, $x \in \mathbb{R}^d$

$$\delta \le \operatorname{tr} a(\omega, p) \le K, \quad |f(\omega, p, x)| \le K.$$

(d) The first- and second-order generalized derivatives of σ with respect to p are bounded by K. The function $f(\omega, p, x)$ satisfies the Lipschitz condition in (p, x) with the constant K and $f(\omega, p, x) - K(|p|^2 + |x|^2)$ is concave in (p, x) for any ω. We have $\|\phi\|_{C^4(\mathbb{R}^d)} \le K$.

(e) For $x \in D$, $w \in \mathbb{S}^d$ we have

$$\begin{aligned} H(w_{ij}, x) &:= \inf_{\omega\in\Omega}[a^{ij}(\omega, 0)w_{ij} + f(\omega, 0, x)] \\ &= \inf_{\omega\in\Omega, p\in\mathbb{R}^{d_0}}[a^{ij}(\omega, p)w_{ij} + f(\omega, p, x)]. \end{aligned}$$

(f) For any unit $\xi \in \mathbb{R}^d$ we have

$$\sup_\omega a^{ij}(\omega, 0)\xi^i\xi^j \ge \delta.$$

We also assume that we are given a function $B(x)$ on $\bar{D}$ with values in $\mathscr{M}_+$ and a function $P(\omega, x)$ on $\Omega \times \bar{D}$ with values in the set of all $d_0 \times d$ matrices.

Define the following operator of differentiation of functions depending on (p, x) along the vector field $(P\xi, \xi) \in \mathbb{R}^{d_0+d}$:

$$\partial(\xi) = \xi^i\left[\frac{\partial}{\partial x^i} + P^{ki}(\omega, x)\frac{\partial}{\partial p^k}\right].$$

We also put $u_{(\xi)} = \xi^i u_{x^i}$. Note that by Assumption 5.1(d) the function σ is continuously differentiable.

Assumption 5.2. (a) $\|P\| \le K$, the function B, and its first- and second-order derivatives are bounded by K.

(b) For $x \in \partial D$, $\xi \perp \psi_x(x)$, $|\xi| = 1$, $\omega \in \Omega$, $p = 0$ we have $(B\xi, \xi) \ge \delta$,

$$a^{ij}[\psi_{x^i x^j} + B_{ij}] \le -\delta + K(a\psi_x, \psi_x), \tag{5.1}$$

(5.2)

$$(B\xi, \xi)a^{ij}\psi_{x^i x^j} + \sum_k(\partial(\xi)\psi_{(\sigma^k)})^2 + 2(B\xi, \sigma^k)\partial(\xi)\psi_{(\sigma^k)} \le -\delta + K(a\psi_x, \psi_x)$$

with $\partial(\xi)\psi_{(\sigma^k)} := \partial(\xi)[\psi_{(\sigma^k)}]$ (and with summation in i, j, k).

Theorem 5.3. *Under the above assumptions there exists a unique function $u \in C(\bar{D}) \cap C^{1,1}(D)$ such that $u = \phi$ on ∂D and $H(u_{x^i x^j}, x) = 0$ (a.e.) in D. Furthermore, the function u and its first and second generalized derivatives in D are bounded by a constant depending only on δ, K, d, d_1, d_0, and*

the diameter of D. If the Bellman equation (4.6) *is quasi-nondegenerate, then $u \in C^{2+\alpha}(\bar{D})$ for some $\alpha \in (0, 1)$.*

The first two assertions of this theorem follow from Theorem 1.1 in [20] if we take there $D_1 = D$, $B_0 = I$, $B_1 = B$, $\psi_1 = \psi$, $\psi_0 = (2R)^2 - |x - x_0|^2$, where R is the diameter of D and x_0 is a point in D. The last assertion follows from Remark 4.9 and from results in [17] (see, for instance, Example 8.2.2 there). The reader can also find there a generalization of the last assertion of the theorem, namely, that nondegeneracy of the Bellman equation in a subdomain of D implies the better smoothness of solutions in the same subdomain.

Remark 5.4. If we want to apply Theorem 5.3 to an example of the Bellman equation with constant coefficients as in Definition 4.4 (with no dependence on p and with $\sigma = \sqrt{2a}$), everything is smooth and bounded, and the number of independent variables cannot be reduced (see Remark 4.8), then the only assumptions to check are Assumption 5.2(b) along with Assumption 5.1(b), (c). It is worth noting (cf. [19]) that if in addition the domain D is strictly convex, then the first assumption is fulfilled automatically. Indeed, in this case the matrix ψ_{xx} can be taken strictly negative on the boundary, and if we take as B a suitable continuation of the function $-(1/2)\psi_{xx}$ from ∂D to $\bar{D}$ (and $P = 0$ if there are any p), then on ∂D we have

$$a^{ij}[\psi_{x^ix^j} + B_{ij}] = \frac{1}{2}a^{ij}\psi_{x^ix^j}, \quad (B\xi, \xi)a^{ij}\psi_{x^ix^j} + \sum_k (\partial(\xi)\psi_{(\sigma^k)})^2$$

$$+ 2(B\xi, \sigma^k)\partial(\xi)\psi_{(\sigma^k)} = -\frac{1}{2}(\psi_{xx}\xi, \xi)a^{ij}\psi_{x^ix^j},$$

where the right-hand sides are strictly negative once $\operatorname{tr} a \geq \delta$.

Theorem 5.3 says that Bellman equation (4.6) is a "good" one if the function $f(\omega, x)$ is Lipschitz continuous in x with a constant independent of ω and if it is such that $f(\omega, x) - K|x|^2$ is concave in x for a constant K independent of ω. It turns out that the concavity of $f(\omega, x) - K|x|^2$ is easy to obtain when the function f can be represented as a composition of a concave and a twice continuously differentiable function. In these cases the following theorem is useful.

Theorem 5.5. *Let $\mathscr{L}$ be a convex set in $\mathbb{R}^n$, and for every $l \in \mathscr{L}$ let an open domain $\Theta(l)$ in $\mathbb{S}^d$ be defined.*

(i) *If $\Theta(l)$ is convex in l in the sense that for any $l_1, l_2 \in \mathscr{L}$, $w_i \in \Theta(l_i)$, we have*

$$\frac{w_1 + w_2}{2} \in \Theta\left(\frac{l_1 + l_2}{2}\right),$$

then for any $\omega \in \mathbb{S}^d$ the function

$$g(\omega, l) = -\inf_{w \in \Theta(l)} \omega_{ij} w_{ij} \tag{5.3}$$

is concave in $\mathscr{L}$. In particular, if it is finite for an $\omega \in \mathbb{S}^d$ and $\mathscr{L}$ has a nonempty interior $\mathscr{L}^o$, then in addition $g(\omega, l)$ satisfies the Lipschitz condition in l on any compact subset of $\mathscr{L}^o$ for the same $\omega \in \mathbb{S}^d$.

(ii) *Take a constant* $K \geq 0$, $l_0, l_1 \in \mathscr{L}$, *and assume that for any* $w \in \partial\Theta(l_0)$ *there is a matrix* $v \in \mathbb{S}^d$ *such that* $||v|| \leq K|l_1 - l_0|$ *and* $w + v \in \bar{\Theta}(l_1)$. *Then for any* $\omega \in \Omega$ *we have*

$$g(\omega, l_0) \leq g(\omega, l_1) + K|l_1 - l_0| \cdot ||\omega||. \tag{5.4}$$

(iii) *If for some* $l_0, l \in \mathscr{L}$ *we have* $\Theta(l) \subset \Theta(l_0)$, *then* $g(\omega, l) \leq g(\omega, l_0)$, *and if for a* $l_0 \in \mathscr{L}$ *and any* $\varepsilon > 0$ *there is a* $l \in \mathscr{L}$ *such that* $l \neq l_0$, $|l_0 - l| \leq \varepsilon$, *and* $\Theta(l) \subset \Theta(l_0)$, *then for any* $\omega \in \Omega$

$$g(\omega, l_0) \geq \liminf_{l \to l_0} g(\omega, l). \tag{5.5}$$

Proof. To prove assertion (i) it suffices to note that for any $l_1, l_2 \in \mathscr{L}$, $w_i \in \Theta(l_i)$ we have

$$\begin{aligned}\omega_{ij}w_{1ij} + \omega_{ij}w_{2ij} = 2\omega_{ij}\frac{w_{1ij} + w_{2ij}}{2} &\geq 2 \inf_{w \in \Theta((l_1+l_2)/2)} \omega_{ij}w_{ij} \\ &= -2g\left(\omega, \frac{l_1 + l_2}{2}\right).\end{aligned}$$

Under conditions in (ii) for $w \in \partial\Theta(l_0)$ we have

$$\omega_{ij}w_{ij} = \omega_{ij}(w_{ij} + v_{ij}) - \omega_{ij}v_{ij} \geq \inf_{w \in \Theta(l_1)} \omega_{ij}w_{ij} - ||\omega|| \cdot ||v||,$$

and this yields (5.4).

Assertion (iii) is obvious since for the point l we have

$$-g(l, \omega) = \inf_{w \in \Theta(l)} \omega_{ij}w_{ij} \geq \inf_{w \in \Theta(l_0)} \omega_{ij}w_{ij} = -g(l_0, \omega).$$

The theorem is proved.

The following theorem allows us to check the conditions of Theorem 5.5 if the domains $\Theta(l)$ are defined with the help of functions on $\mathbb{S}^d$. This theorem plays a crucial role in connecting branches (2.10) with Bellman equations. To state the theorem we need the following

Assumption 5.6. We are given a convex set $\mathscr{L}$ in $\mathbb{R}^n$ with nonempty interior $\mathscr{L}^o$ and an open convex cone $C \subset \mathbb{S}^d$ with vertex at the origin containing $\mathscr{M}_+^o$ and such that $C \neq \mathbb{S}^d$. On $C \times \mathscr{L}$ we are given a finite continuous function $F(w, l)$ which is convex in w. Define

$$\Theta(l) = \{w \in C : F(w, l) < 1\}.$$

For a number t_1 and any $w \in C$, $l \in \mathscr{L}$ it holds that

$$\lim_{t \to \infty} F(tw, l) < 1, \quad F(t_1 I, l) \leq 1. \tag{5.6}$$

Theorem 5.7. *Let Assumption* 5.6 *be satisfied. Then*

(i) *The sets* $\Theta(l)$ *are convex, and for any* $l \in \mathscr{L}$ *the equation* $(u_{x^ix^j}) \in \partial\Theta(l)$ *is elliptic.*

(ii) *For* $w \in \mathbb{S}^d$, $l \in \mathscr{L}$

$$w \in \partial\Theta(l) \iff H(w, l) := \inf_{\omega\in\Omega}(\omega_{ij}w_{ij} + g(\omega, l)) = 0, \tag{5.7}$$

where

(5.8)
$$\Omega = \{\omega \in \mathscr{M}_+ : \operatorname{tr}\omega = 1,\ \operatorname{tr}\omega w \geq 0\ \forall w \in C\}, \qquad g(\omega, l) = -\inf_{w\in\Theta(l)} \omega_{ij}w_{ij}.$$

Moreover, $-t_1 \leq g(\omega, l) \leq 0$.

(iii) *If for any* (w_1, l_1), $(w_2, l_2) \in C \times \mathscr{L}$ *we have*

$$F\left(\frac{w_1 + w_2}{2}, \frac{l_1 + l_2}{2}\right) \leq \max(F(w_1, l_1), F(w_2, l_2))$$

(quasiconvexity of $F(w, l)$ *in* (w, l)*), then the function* $g(\omega, l)$ *is concave in* l *and, in particular, on any compact subset of* $\mathscr{L}^o$ *satisfies a Lipschitz condition in* l *with a constant independent of* ω.

Proof. Assertion (i) follows from Theorem 2.10 since the sets $\Theta(l)$ are open and obviously convex and since by virtue of (5.6) for any $w \in \mathscr{M}_+^o \subset C$ we have $tw \in \Theta(l)$ if t is large enough.

(ii) The equivalence in question follows from Theorem 4.11. We get the inequalities $-t_1 \leq g(\omega, l) \leq 0$ from Theorem 4.11 and from the second condition in (5.6), which implies that $t_1 I \in \bar{\Theta}(l)$.

(iii) Observe that quasiconvexity of F implies the convexity of $\Theta(l)$, and we get the concavity of g in l from Theorem 5.5, which along with its boundedness gives an estimate of the Lipschitz constant. The theorem is proved.

Remark 5.8. Sometimes the case in which the domains $\Theta(l)$ are defined with the help of concave functions $F(w, l)$ can be reduced to the one considered in Theorem 5.7 if we notice that $F^{-1}(w, l)$ is convex for positive concave F.

Let us apply this theorem along with Theorem 5.3 in the proof of one more version of Theorem 1.1.

Theorem 5.9. *Under the hypotheses of Theorem* 1.1 *assume that for any* $k = 0, \dots, m-1$ *the function* l_k *is either strictly positive in* $\bar{D}$ *or identically zero. Then the assertions of this theorem hold true.*

We will use simple

Lemma 5.10. *If a function* $F(w)$ *is convex and homogeneous of degree* $-\alpha < 0$ *in a cone* $C \subset \mathbb{S}^d$, *then the function* $l^{\alpha+1}F(w)$ *is a convex function of* (w, l) *in* $C \times R_+$.

Proof. Bearing in mind obvious approximations we can confine ourselves to the case of smooth F. In this case for $w \in C$ we have

$$F(w) = \sup_{v\in C}(F_{v_{ij}}(v)w_{ij} + F(v) - F_{v_{ij}}(v)v_{ij}) = \sup_{v\in C}(F_{v_{ij}}(v)w_{ij} + (\alpha+1)F(v)),$$

where we applied the Euler theorem about homogeneous functions. Multiplying the extreme terms by $l^{\alpha+1}$ and replacing v by vl we get

$$l^{\alpha+1}F(w) = \sup_{v\in C}(F_{v_{ij}}(v)w_{ij} + l(\alpha+1)F(v)),$$

which is indeed convex in (w, l) as an upper envelope of linear functions. The lemma is proved.

Proof of Theorem 5.9. For simplicity of notation we assume that $\varepsilon^{-1} > l_0, \dots, l_{n-1} > \varepsilon$ in $\bar{D}$, where the constant $\varepsilon > 0$, and $l_n, \dots, l_{m-1}$ identically equal zero (if $n = 0$ we assume that all $l_k \equiv 0$). We could achieve this by taking an appropriate ε and by renumbering P_k since we do not rely on any specific properties of them depending on k.

For $l \in \mathscr{L} = (\varepsilon, \varepsilon^{-1})^n$ take C_m from Lemma 4.16 and define

(5.9)
$$C = C_m, \quad F(w, l) = \sum_{k=0}^{n-1} l_k^{m-k+1} \frac{P_k}{P_m}(w), \quad \Theta(l) = \{w \in C : F(w, l) < 1\}$$

(if $n = 0$, by definition $F \equiv 0$). With these objects Assumption 5.6 is satisfied due to obvious reasons and to Lemma 4.16. By using Theorem 5.7 and formula (4.11) we see that our theorem, actually, relates to the Bellman equation

$$H(u_{x^i x^j}, l(x)) = 0, \tag{5.10}$$

where H is taken from (5.7). Below we also use the objects Ω, $g(\omega, l)$ from Theorem 5.7.

By Lemmas 5.10 and 4.16 the function $F(w, l)$ is convex in (w, l) and, in particular, quasi-convex. Theorem 5.7(iii) says that the function $g(\omega, l)$ is uniformly bounded and concave in $l \in \mathscr{L}$. Next, for unit $\xi \in \mathbb{R}^d$ in any reasonable sense

(5.11)
$$(g(\omega, l(x)))_{x^i x^j}\xi^i\xi^j = g_{l_k}(\omega, l(x))l_{kx^ix^j}\xi^i\xi^j + g_{l_k l_r}(\omega, l(x))l_{k(\xi)}l_{r(\xi)},$$

where the last term is negative and the first is bounded since the range of $l(x)$, $x \in \bar{D}$, is a compact subset of the open set $\mathscr{L}$. It follows easily that for the function $f(\omega, x) := g(\omega, l(x))$, which is uniformly bounded in $\Omega \times \mathbb{R}^d$, there exists a constant N such that $f(\omega, x) - N|x|^2$ is concave in x for any $\omega \in \Omega$. It is also seen that $f(\omega, x)$ satisfies the Lipschitz condition in x with a constant independent of ω.

Naturally, we want to apply Theorem 5.3 to equation (5.10). So far we do not introduce parameters p, and we put $a(\omega) = \omega$ and $\sigma = \sqrt{2a}$. Next, take a constant $t > 0$ to be specified later, and near ∂D let $\bar{\psi}(x) = \operatorname{dist}(x, \partial D)$ if $x \in \bar{D}$, $\bar{\psi}(x) = -\operatorname{dist}(x, \partial D)$ if $x \notin D$, $\psi = \bar{\psi} - t\bar{\psi}^2$ and continue ψ in an appropriate manner to satisfy Assumption 5.1(a).

We know that the normal second-order derivative of ψ on ∂D is $-2t$ and that the matrix of its tangential second-order derivatives has eigenvalues $-\kappa^1, \dots, -\kappa^{d-1}$. Therefore, for large t on ∂D we have

$$P_m(-\psi_{xx}) \sim 2tP_{m-1,d-1}(\kappa^1, \dots, \kappa^{d-1}),$$

which is strictly positive. We fix t such that $P_m(-\psi_{xx}) > 0$ on ∂D. According to a nice observation from [5, p. 274], there is at least one point on ∂D at which all κ_i are nonnegative. At this point $-\psi_{xx} \in \bar{C}_m$, and since $P_m(-\psi_{xx})$ is strictly positive on the connected set ∂D, we have $-\psi_{xx} \in C_m$ not only at

this point but everywhere on ∂D. The set of all values of $-\psi_{xx}(x)$, $x \in \partial D$, is a compact subset of C_m, which by definition (5.8) of Ω implies that for a $\delta > 0$ Assumption 5.1(b) is satisfied (recall that $a(\omega) = \omega$). The nontrivial parts in statements (c) and (d) of this assumption have been checked above.

The weak nondegeneracy of (5.10) (Assumption 5.1(f)) follows from the last statement of Theorem 4.11 as in the proof of Theorem 4.15.

Finally, the function $f(\omega, x)$ is uniformly continuous in $x \in \mathbb{R}^d$ uniformly in ω, so that the function $H(w, l(x))$ is continuous on $\mathbb{S}^d \times \bar{D}$. Furthermore, in the proof of Theorem 4.15 we saw that if l_k are constant and $\sum l_k > 0$, then the equation (2.2) is quasi-nondegenerate. By Remark 4.6 this yields the nondegeneracy of the corresponding Bellman equation. The continuity of $H(w, l(x))$ along with Remark 4.7 allows us to affirm that Bellman equation (5.10) is quasi-nondegenerate under the condition that $\sum l_k > 0$ in $\bar{D}$.

We conclude that if Assumption 5.2 was satisfied, the assertions of our present theorem would follow directly from Theorem 5.3.

Of course, if D were strictly convex or close to a strictly convex domain, we could refer to Remark 5.4 without appealing to any kind of parameters. But in the general case we do not know if it is possible not to use parameters in checking Assumption 5.2. The way to introduce appropriate parameters p (say, such that Assumption 5.1(e) is satisfied) is prompted by the observation that the main term $P_m(u_{x^i x^j})$ of our equation (1.5) is invariant under any orthogonal transformation in $\mathbb{R}^d$, which is reflected in the relations $e^p C_m e^{-p} = C_m$, $e^p \Omega e^{-p} = \Omega$ valid for any skew-symmetric $d \times d$ matrix p.

That is why we revise the above arguments, introducing $\mathbb{R}^{d_0}$ as the space of all skew-symmetric matrices and taking this time

$$\sigma(\omega, p) := e^p \sigma(\omega) = e^p \sqrt{2\omega}, \quad f(\omega, p, x) = f(\omega, x).$$

Now observe that $a(\omega, p) = e^p \omega e^{-p}$, and since $e^p \Omega e^{-p} = \Omega$, Assumption 5.1(e) is indeed satisfied. The same argument shows that we need not repeat checking the other requirements in Assumption 5.1. Therefore, to finish the proof it remains again to check Assumption 5.2, or if we take there $B := t(\delta_{ij})$ with a constant $t > 0$, then we need only to find a t and a uniformly bounded function $P(\omega, x)$ with values in the space of linear operators acting from $\mathbb{R}^d$ into $\mathbb{R}^{d_0}$ such that for some constants K, $\delta > 0$ and all $x \in \partial D$, $\xi \perp \psi_x(x)$, $|\xi| = 1$, $\omega \in \Omega$ inequalities (5.1) and (5.2) are satisfied, where

$$\partial(\xi)\psi_{(\sigma^k)}(x) = \frac{d}{dh}\psi_{x^i}(x + h\xi)\sigma^{ik}(\omega, hP(\omega, x)\xi)|_{h=0}.$$

(We prefer a different representation for the same operator $\partial(\xi)$, because the space $\mathbb{R}^{d_0}$ is a matrix space and the original formula looks confusing.)

We find that

$$\partial(\xi)\psi_{(\sigma^k)}(x) = \psi_{x^i(\xi)}(x)\sigma^{ik}(\omega) + \psi_{x^i}(x)[P(\omega, x)\xi]_{ir}\sigma^{rk}(\omega).$$

Now as in [18] we define $P(\omega, x)\xi = P(x)\xi$ by the formula

$$[P\xi]_{ir} = \psi_{x^i(\xi)}\psi_{x^r} - \psi_{x^i}\psi_{x^r(\xi)}.$$

Notice that on ∂D we have $|\psi_x|^2 = 1$, so that $\psi_x \perp \psi_{x(\xi)}$ for our $\xi \perp \psi_x$. Therefore,

$$\psi_{x^i}(x)[P(\omega, x)\xi]_{ir} = -\psi_{x^r(\xi)}, \quad \partial(\xi)\psi_{(\sigma^k)} = 0,$$

which along with Assumption 5.1(b), which has already been checked, implies that inequality (5.2) with $K = 0$ holds for any constant t, whereas to satisfy (5.1) it suffices to take the constant t small enough.

We finally succeeded in verifying the assumptions of Theorem 5.3, and thus our theorem is proved.

Remark 5.11. In this proof we needed the specific assumption that either l_k is strictly positive or identically zero, only to be able to use relations like (5.11) to derive that the function

$$f(\omega, x) = g(\omega, l_0^+(x), \dots, l_{n-1}^+(x))$$

is Lipschitz continuous in x and that $f - N|x|^2$ is concave for a constant N, by using the fact that $g(\omega, l)$ is concave in $\mathscr{L}$ and, in particular, has bounded first derivatives in l on any compact subset of $\mathscr{L}$. Approximating f by $g(\omega, \varepsilon + l_0^+, \dots, \varepsilon + l_{n-1}^+)$, we easily see that if $\mathscr{L}$ has the form, say, $[0, M)^n$,and $l(x)$ run through all of $\mathscr{L}$, then to get these properties of f, along with concavity and *continuity* of $g(\omega, l)$ in $\mathscr{L}$,we need in addition an estimate of its first derivatives with respect to l in $\mathscr{L}^o$ and the inequalities $g_{l_i} \le 0$, $i = 1, \dots, n$, to be true.

It turns out that we can obtain these additional properties of first derivatives of g without using the convexity of g but instead relying on two last general statements of Theorem 5.5, versions of which for the situation of Theorem 5.7 we present in the following theorem. In this theorem we also discuss a nondegeneracy condition for Bellman equations.

Theorem 5.12. *Let Assumption* 5.6 *be satisfied, and assume the notation from Theorem* 5.7. *We assert that*

(i) *If for some* $l_0, l \in \mathscr{L}$ *we have* $F(w, l) \ge F(w, l_0)$ *for all* $w \in C$, *then* $g(\omega, l) \le g(\omega, l_0)$ *for any* $\omega \in \Omega$.

(ii) *If* (a) *for any* $l \in \mathscr{L}^o$ *we have* $\bar{\Theta}(l) \subset C$; (b) *the function* F *is differentiable in* $C \times \mathscr{L}^o$ *and for a constant* K *we have*

$$|F_l(w, l)| < -K \sum_{i=1}^{d} F_{w_{ii}}(w, l),$$

provided that $(w, l) \in C \times \mathscr{L}^o$ *and* $F(w, l) = 1$; *and* (c) *for any* $l_0 \in \mathscr{L}$ *and any* $\varepsilon > 0$ *there is a* $l \in \mathscr{L}$ *such that* $l \ne l_0$, $|l_0 - l| \le \varepsilon$, *and* $F(w, l) \ge F(w, l_0)$ *for all* $w \in C$,

then $|g(\omega, l_1) - g(\omega, l_2)| \le K|l_1 - l_2|$ *for any* $\omega \in \Omega$, $l_i \in \mathscr{L}$.

(iii) *If* (d) *the function* $g(\omega, l)$ *is uniformly continuous with respect to* l *uniformly in* ω, (e) *the set* $\mathscr{L}$ *is closed and bounded, and* $\bar{\Theta}(l) \subset C$ *for any* $l \in \mathscr{L}$; *and* (f) *there is no* $l \in \mathscr{L}$, $w \in C$, $\xi \in \mathbb{R}^d$ *such that* $F(w + t\xi\xi^*, l) = 1$ *for all* $t \ge 0$,

then for the function H *from* (5.7) *and for any constant* $N > 0$ *there exists a* $\delta > 0$ *such that*

$$\begin{aligned} &\frac{\partial^+}{\partial t} F(w + t\xi\xi^*, l)|_{t=0} \le -\delta, \\ &\frac{\partial^+}{\partial t} H(w + t\xi\xi^*, l)|_{t=0} \ge \delta \end{aligned} \tag{5.12}$$

whenever $||w|| \le N$, $l \in \mathscr{L}$, $H(w, l) = 0$, $|\xi| = 1$ (*or equivalently, whenever* $||w|| \le N$, $w \in C$, $l \in \mathscr{L}$, $F(w, l) = 1$, $|\xi| = 1$).

Proof. Assertion (i) is trivial because $\Theta(l) \subset \Theta(l_0)$.

(ii) As we mentioned above we apply the second and third assertions of Theorem 5.5. First note that by the continuity of F we have $F(w, l) < 1$ if $F(w, l_0) < 1$ and l is close enough the l_0. Therefore, $g(\omega, l)$ is lower semicontinuous. This along with assumption (c), where $\Theta(l) \subset \Theta(l_0)$, and with Theorem 5.5(iii) gives us equality in (5.5) and allows us to investigate $g(\omega, l)$ for $l \in \mathscr{L}^o$ only.

Note that by (a) for any $l \in \mathscr{L}^o$ we have

$$\partial\Theta(l) = \{w \in C : F(w, l) = 1\}. \tag{5.13}$$

Now we take $l_0, l_1 \in \mathscr{L}^o$, $w_0 \in \partial\Theta(l_0)$ and claim that for any $t \in [0, 1]$ we have $F(w_t, l_t) \le 1$ where $w_t = w_0 + tK|l_1 - l_0|I$, $l_t = l_0 + t(l_1 - l_0)$. To establish this it actually suffices to show that if $t_0 \in [0, 1]$, $l_0 \ne l_1$, and $F(w_{t_0}, l_{t_0}) = 1$, then $p(t_0) < 0$, where $p(t) = dF(w_t, l_t)/dt$. But $l_{t_0} \in \mathscr{L}^o$ and $w_{t_0} \in \partial\Theta(l_{t_0})$. Therefore, from (b) we see that indeed

$$p(t_0) = K|l_1 - l_0| \sum_{i=1}^{d} F_{w_{ii}}(w_{t_0}, l_{t_0}) + F_{l^k}(w_{t_0}, l_{t_0})(l_1 - l_0)^k < 0.$$

Thus, for $v = K|l_1 - l_0|I$ we have $F(w_0 + v, l_1) \le 1$, and the assertion in (ii) follows from Theorem 5.5(ii).

To prove (iii) we actually repeat the corresponding part of the proof of Theorem 5.9. First, observe that by virtue of (d) the function $H(w, l)$ is continuous in (w, l). Therefore, the set $\{(w, l) : ||w|| \le N, H(w, l) = 0\}$ is closed and bounded. As in Remark 4.7 we conclude that to prove the second inequality in (5.12) we must only show that its left-hand side is strictly positive for any given w, l, ξ such that $H(w, l) = 0$, $|\xi| = 1$. But this can be done exactly as in Remark 4.6 since by Theorem 5.7(ii) we have $w \in \partial\Theta(l)$, from Theorem 5.7 (i) we know that $w + t\xi\xi^* \in \bar{\Theta}(l)$ for $t > 0$, and (5.13) along with (f) leaves the only possibility that $w + t\xi\xi^* \in \Theta(l)$ for a (actually, for all) $t > 0$. The first inequality in (5.12) can be proved quite similarly with the only difference that $F(w + t\xi\xi^*, l) < 1$ for $t > 0$ and $H(w + t\xi\xi^*, l) > 0$ for $t > 0$. The theorem is proved.

Before we show the application of the above theorem to the proof of Theorem 1.1 we state the following lemma, which will be proved in Section 6. We use the notation of Lemma 4.16.

Lemma 5.13. (i) *For any* $m = 1, \dots, d$, $k = 0, 1, \dots, m$, $w \in C_m$, *the function*

$$\frac{P_m^{1/m}}{P_k^{1/k}}(w + tI)$$

is a nondecreasing function of t *on* $[0, \infty)$.

(ii) *If we normalize* P_k *with the help of constant factors so as to have that* $P_k(I) = 1$, *then the function* $\log P_k(w)$ *is a concave function on the integers* $k = 0, 1, \dots, m$ *for any* $w \in C_m$.

Proof of Theorem 1.1. Define M as the maximum of $l_k^+(x)$ over k, x, and let $\mathscr{L} = [0, M+1)^m$. Next, take C_m from Lemma 4.16, for $n = m$ assume notation (5.9) and the notation Ω, $g(\omega, l)$ from Theorem 5.7.

As we explained in Remark 5.11, we must only show that $g(\omega, l)$ decreases in every l_i and satisfies the Lipschitz condition in $l \in \mathscr{L}$ with a constant independent of ω. Obviously, $F(w, l)$ increases in each l_i, which by Theorem 5.12(i) yields that g is decreasing indeed.

By the same theorem, to show that g is Lipschitz continuous in $l \in \mathscr{L}$ it suffices to check conditions (a), (b), (c) in its assertion (ii). It turns out that it is more natural to prove a more general result that the function $g(\omega, l^\alpha)$, where $\alpha_k = (m-k)/(m-k+1) < 1$ and $l^\alpha = (l_0^{\alpha_0}, \dots, l_{m-1}^{\alpha_{m-1}})$, is Lipschitz continuous in l with a constant independent of ω (and M). Of course, the function $g(\omega, l^\alpha)$ corresponds to $F(w, l^\alpha)$ in the same way as $g(\omega, l)$ corresponds to $F(w, l)$. Therefore, we can confine ourselves to checking conditions of Theorem 5.12(ii) for the function $F(w, l^\alpha)$ instead of $F(w, l)$.

Condition (a) has been checked in the proof of Theorem 4.15 (see (4.12)). Condition (c) is satisfied since F decreases in l_i. To check (b) we have to find a constant $\varepsilon > 0$ such that

(5.14)
$$\varepsilon \sum_{k=0}^{m-1} (m-k) l_k^{m-k-1} \frac{P_k}{P_m}(w) < \sum_{i=1}^{d} \sum_{k=0}^{m-1} l_k^{m-k} \frac{P_k}{P_m} \left\{ \frac{P_{mw_{ii}}}{P_m} - \frac{P_{kw_{ii}}}{P_k} \right\}(w),$$

whenever $w \in C_m$, $l_k \geq 0$ and

$$\sum_{k=0}^{m-1} l^{m-k} \frac{P_k}{P_m} = 1. \tag{5.15}$$

For the sake of convenience let us deal with normalized functions P_k. Then

$$P_k(w) = \frac{k!}{d!} \frac{d^{d-k}}{dt^{d-k}} \det(w + tI)|_{t=0},$$

$$\sum_{i=1}^{d} P_{kw_{ii}}(w) = \frac{d}{dt} P_k(w + tI)|_{t=0} = kP_{k-1}(w).$$

Next, from Lemma 5.13(i) we see that

$$\frac{1}{m}\sum_{i=1}^{d}\frac{P_{mw_{ii}}}{P_m} \geq \frac{1}{k}\sum_{i=1}^{d}\frac{P_{kw_{ii}}}{P_k},$$

$$\sum_{i=1}^{d}\left\{\frac{P_{mw_{ii}}}{P_m} - \frac{P_{kw_{ii}}}{P_k}\right\} \geq \sum_{i=1}^{d}\left(1-\frac{k}{m}\right)\frac{P_{mw_{ii}}}{P_m} = (m-k)\frac{P_{m-1}}{P_m}.$$

From this and from condition (5.15) it follows that the right-hand side in (5.14) is not less than P_{m-1}/P_m.

On the other hand, from (5.15) we get

$$l_k^{m-k-1}\left[\frac{P_k}{P_m}\right]^{(m-k-1)/(m-k)} \leq 1, \quad l_k^{m-k-1}\frac{P_k}{P_m} \leq \left[\frac{P_k}{P_m}\right]^{1/(m-k)},$$

and by Lemma 5.13(ii)

$$\frac{1}{m-k}\log P_k + (1-\frac{1}{m-k})\log P_m \leq \log P_{m-1}, \quad \left[\frac{P_k}{P_m}\right]^{1/(m-k)} \leq \frac{P_{m-1}}{P_m},$$

$$l_k^{m-k-1}\frac{P_k}{P_m} \leq \frac{P_{m-1}}{P_m}.$$

We conclude that (for our normalized P_k) inequality (5.14) holds with $\varepsilon = (2m)^{-1}$, and the theorem is proved.

Remark 5.14. We have used that the functions l_k enter equation (1.5) like $(l_k^+)^{m-k+1}$ and not like $(l_k^+)^{m-k}$ only in one place, namely, in the proof of Theorem 5.9 when while checking condition (iii) of Theorem 5.7 we have used Lemma 5.10.

It turns out that given $n \leq m-1$, the assertions of Theorem 1.1 continue to hold true if $l_k \equiv 0$ for $k \neq n$ and if we replace $(l_n^+)^{m-n+1}$ by $(l_n^+)^{m-n}$. The reason for this is that for the function $F(w, l) = l^n F(w) = l^n P_n/P_m$ by its homogeneity and convexity in w, for any $w_i \in C_m$ and $l_i > 0$ we get

$$\begin{aligned} F\left(\frac{w_1+w_2}{2}, \frac{l_1+l_2}{2}\right) &= F\left(\frac{w_1+w_2}{l_1+l_2}\right) \\ &\leq \frac{l_1}{l_1+l_2}F\left(\frac{w_1}{l_1}\right) + \frac{l_2}{l_1+l_2}F\left(\frac{w_2}{l_2}\right) \\ &\leq \max_{i=1,2} F\left(\frac{w_i}{l_i}\right) = \max_{i=1,2} F(w_i, l_i). \end{aligned}$$

In particular, we see that sometimes it is convenient to use exactly the quasi-convexity condition in assertion (iii) of Theorem 5.7 and not the convexity of F in both arguments.

Remark 5.15. In the above proof of Theorem 1.1 we could show the nondegeneracy of equation (1.5), when $\max_k l_k > 0$ in $\bar{D}$, by using the last assertion of Theorem 5.12 *where* for a constant $\varepsilon > 0$ we take $\mathscr{S} = [0, M]^m \cap \{\sum l_k \geq \varepsilon\}$.

Remark 5.16. In our proof of Theorem 1.1 we used the assumption $l \in C^2(\mathbb{R}^d)$ only to show that there is a constant N such that $f - N|x|^2$ is concave. But

formulas like (5.11) show that for this it is quite sufficient to assume that for a constant K and any $k = 0, 1, \dots, m-1$ the functions $l_k(x) + K|x|^2$ are convex. By the way, if we assume only this, then there is no need to consider positive parts of l_k since they have the same property. Instead we can require that l_k's be nonnegative.

Remark 5.17. Instead of the assumption $l \in C^2(\mathbb{R}^d)$, in Theorem 1.1 assume that for a constant K and any $k = 0, 1, \dots, m-1$ the functions $l_k(x) + K|x|^2$ are convex, that for a $\beta \in (0, 1)$ we have $l_k \in C^{1+\beta}(\mathbb{R}^d)$, and that $\max_k l_k > 0$ in $\bar{D}$. Then the solution u lies in $C^{3+\beta}(\bar{D})$.

Indeed, we use Remark 5.16 and note that under our hypotheses in notation (5.9) with $n = m$ the solution u satisfies the equation $F(u_{x^i x^j}, l(x)) = 1$, and the function $F(w, l(x))$ is of class $C^{1+\beta}(\mathbb{R}^d)$. Moreover, from (5.12) we conclude that the equation $F(u_{x^i x^j}, l(x)) = 1$ is nondegenerate on the solution u. Therefore, our assertion follows in a well-known way from the theory of *linear* elliptic equations.

Of course, if $\max_k l_k > 0$ in a subdomain of D rather than in $\bar{D}$, then we get regularity (or better regularity) of u in this subdomain.

6. Tests for convexity of Θ

We have seen in the preceding sections that the convexity of an open component of the set $\{w : F(w) > 0\}$ makes the investigation of equation (1.1) much easier. Here we want to describe some cases when this convexity can be deduced from general properties of F. We also give proofs of Lemmas 4.16 and 5.10.

The following general Bochner theorem (see, for instance, [12, p. 43], may be useful.

Theorem 6.1. *If $F(x)$ is an entire function on $\mathbb{R}^d$, then all components of the set $\{x : F(x + iy) \neq 0 \ \forall y \in \mathbb{R}^d\}$ are convex.*

In the sequel we consider a real polynomial Q_m of degree $m \geq 1$ of the real variables $\lambda^1, \dots, \lambda^d$. We write it as $Q_m(\lambda)$, where $\lambda = (\lambda^1, \dots, \lambda^d) \in \mathbb{R}^d$.

Theorem 6.2. *Define Θ as an open connected component of the set $\{\lambda : Q_m(\lambda) > 0\}$, and let $\Theta \neq \emptyset$. If for all $\mu \in \Theta$, $\lambda \in \mathbb{R}^d$ the polynomial $Q_m(\mu + t\lambda)$ as a polynomial in t has only real zeros, then Θ is convex and, moreover, the function $Q_m^{1/m}$ is concave on Θ.*

This elementary fact is well known (see, for instance, [18]). Actually, the convexity of Θ follows from the Bochner theorem or from the concavity of $Q_m^{1/m}$, and the latter, one obtains by using the Cauchy inequality after differentiating twice with respect to t the formula

$$Q_m^{1/m}(\mu + t\lambda) = Q_m^{1/m}(\lambda)[(t - t_1) \cdot \dots \cdot (t - t_m)]^{1/m},$$

where t_i are roots of $Q_m(\mu + t\lambda)$.

This theorem advances the problem of describing polynomials satisfying its hypotheses. It turns out that a subclass of these polynomials is very well known in the theory of hyperbolic equations and this is the class of so-called hyperbolic polynomials (see, for instance [8], [1] or [11, Sections 8.7, 12.4]). We need and

prove below some additional properties which the author could not find in the literature.

Below we consider only *homogeneous* polynomials Q_m.

Theorem 6.3. *Fix* λ_0, μ_0, $\nu_0 \in \mathbb{R}^d$, *let* $Q_m(\nu_0) \neq 0$; *and let the polynomial* $Q_m(\nu_0 + t\lambda_0)$ *have exactly* m distinct *real roots (in particular,* $Q_m(\lambda_0) \neq 0$). *Moreover, assume that for every* $s \in R$ *the polynomial* $Q_m(\mu_0 + s\nu_0 + t\lambda_0)$ *as a polynomial in* t *has* m distinct *real roots. Finally, assume that for* $s = 0$ *all these roots are strictly negative. Then for any* $t \geq 0$ *the polynomial* $Q_m(\mu_0 + s\nu_0 + t\lambda_0)$ *as a polynomial in* s *has* m *distinct real roots* $s_1(t) < \cdots < s_m(t)$ *and* $s_i(t)(s_i(t))' > 0$ *for* $t \geq 0$ *and* $i = 1, \dots, m$.

Proof. Denote by $t_1(s), \dots, t_m(s)$ the roots of the equation

$$Q_m(\mu_0 + s\nu_0 + t\lambda_0) = 0. \tag{6.1}$$

Since $t_1(s), \dots, t_m(s)$ are distinct, we can assume that $t_1(s) < \cdots < t_m(s)$. For any i the ratio $T_i(s) := t_i(s)/s$ is a root of $Q_m(\mu_0/s + \nu_0 + T\lambda_0)$. It follows that as $s \to \infty$ the function $T_i(s)$ is bounded and approaches one of m roots $T_1, \dots, T_m$ of the equation $Q_m(\nu_0 + T\lambda_0) = 0$. By our hypotheses $T_1, \dots, T_m$ are distinct and different from zero. Let $T_1 < \cdots < T_m$. Observe that these roots are simple, and in particular, at any T_i the derivative of $Q_m(\nu_0 + T\lambda_0)$ with respect to T is different from zero. Therefore, for μ close to zero the equation $Q_m(\mu + \nu_0 + T\lambda_0) = 0$ also has m real roots, and this implies that $t_i(s)/s \to T_i$ as $s \to \infty$. In the same way for $s \to -\infty$ we get $t_i(s)/s \to T_{m-i+1}$. Notice also that the derivative of $Q_m(\mu_0 + s\nu_0 + t\lambda_0)$ with respect to t at any root is different from zero, and therefore, functions $t_i(s)$ are differentiable.

Now let n be the number of strictly positive T_i. Then there are n functions $t_i(s)$ on $[0, \infty)$ which start from a negative value for $s = 0$ and tend to ∞ as $s \to \infty$ and there are $m - n$ functions $t_i(s)$ on $(-\infty, 0]$ which start from a negative value for $s = 0$ and tend to ∞ as $s \to -\infty$.

It follows that for all $t_0 \geq 0$, above the half line $\{s \geq 0\}$ there will be at least n *different* ($t_i(s)$ are distinct) intersections of graphs of functions $t = t_i(s)$, $i \leq n$, with the straight line $t = t_0$, and above the half line $\{s \leq 0\}$ there will be at least $m - n$ such intersections with graphs of functions $t = t_i(s)$, $i \leq m - n$. Any such intersection gives a root of $Q_m(\mu_0 + s\nu_0 + t_0\lambda_0)$; and since there can be at most m roots, we conclude that for any $t_0 \geq 0$ all roots of $Q_m(\mu_0 + s\nu_0 + t_0\lambda_0)$ are indeed real and distinct.

Moreover, by the same reasoning any line $t = t_0$ with $t_0 \geq 0$ can intersect the graphs of a function $t = t_i(s)$, $s \geq 0$, $i \leq n$, only once, which shows that any such function increases. Of course, the functions $t = t_i(s)$, $s \leq 0$, $i \leq m - n$, decrease. Obviously, $s = s_1(t), \dots, s = s_m(t)$, $t \geq 0$, are inverse functions to $t = t_i(s)$, $s \geq 0$, $i \leq n$, and to $t = t_i(s)$, $s \leq 0$, $i \leq m - n$. From the above argument it follows that for $t \geq 0$ and $i \leq n$ the functions $s_i(t) < 0$ and they decrease, and for $t \geq 0$ and $i \geq m - n$ the functions $s_i(t) > 0$ and they increase. These functions are differentiable in t since for $s = s_i(t)$ and $t \geq 0$ the derivative of $Q_m(\mu_0 + s\nu_0 + t\lambda_0)$ with respect to s does not vanish ($s_i(t)$ are simple roots). Furthermore, $s_i'(t) \neq 0$ since $s_i(t)$ is an inverse of a differentiable function, which along with the above properties of $s_i(t)$ proves our last assertion that $s_i s_i' > 0$. The theorem is proved.

Theorem 6.4. *Fix $\lambda_0 \in \mathbb{R}^d$, and assume that $Q_m(\lambda_0) > 0$, and that for any $\mu \in \mathbb{R}^d$ all roots of the polynomial $Q_m(\mu + t\lambda_0)$ are real. Define Λ_m as the open connected component containing λ_0 of the set $\{\lambda : Q_m(\lambda) > 0\}$.*

Take $\lambda_1 \in \Lambda_m$ and define

$$Q_{m-1}(\lambda) = \frac{\partial}{\partial t} Q_m(\lambda + t\lambda_1)|_{t=0}.$$

Then

(i) *Λ_m is an open convex cone. Moreover, $Q_{m-1}(\lambda) > 0$ on Λ_m so that the open connected component C_{m-1}, containing λ_0, of the set $\{\lambda : Q_{m-1}(\lambda) > 0\}$ contains Λ_m, and for any $\mu \in \mathbb{R}^d$ all roots of the polynomial $Q_{m-1}(\mu + t\lambda_0)$ are real.*

(ii) *The following functions are concave on Λ_m:*

$$Q_m^{1/m}, \quad \log Q_m, \quad \frac{Q_m(\lambda)}{Q_{m-1}(\lambda)}, \quad \log \frac{Q_m(\lambda)}{Q_{m-1}(\lambda)}, \tag{6.2}$$

and the third one tends to zero as $\lambda \to \mu \in \partial\Lambda_m$, $\lambda \in \Lambda_m$.

Proof. (i) Observe that our polynomial Q_m is a so-called hyperbolic λ_0 polynomial and there exists a short and very nice theory of these polynomials (see [8], [1], or [11, Sections 8.7, 12.4]). In (i) we simply collected several results from this theory.

(ii) It is also known that for any $\mu \in \Lambda_m$, $\nu \in \mathbb{R}^d$ all roots of the polynomial $Q_m(\mu + t\nu)$ are real and if $\nu \in \Lambda_m$, they are strictly negative. Therefore, we get our assertion about the first function in (6.2) from Theorem 6.2. The second and the last ones are concave as logarithms of concave functions. Furthermore, as $\lambda \to \mu \in \partial\Lambda_m$, $\lambda \in \Lambda_m$, the functions $\log Q_m(\lambda + t\lambda_1)$ which are concave functions of $t \in (0, \infty)$ tend to a concave function $\log Q_m(\mu + t\lambda_1)$ which is finite on $(0, \infty)$ ($Q_m^{1/m}(\mu + t\lambda_1)$ is concave nonnegative and not identically zero for $t > 0$) and tends to $-\infty$ as $t \downarrow 0$. It follows that the derivative of $\log Q_m(\lambda + t\lambda_1)$ in t at $t = 0$ tends to infinity, and this yields the last assertion in (ii).

It remains to prove the concavity of the third function in (6.2). We need one more known result (see [22]) that any real hyperbolic polynomial can be approximated by polynomials $\bar{Q}_m$ of the same degree m and such that $\bar{Q}_m(\nu + t\lambda_1)$ has m *distinct* real roots unless ν is proportional to λ_1. We can concentrate only on such polynomials, and therefore we assume that $Q_m(\nu + t\lambda_1)$ has m *distinct* real roots unless ν is proportional to λ_1. To start we assume also that $d \geq 3$.

For $t \geq 0$ and fixed $\mu \in \Lambda_m$, $\nu \in \mathbb{R}^d$, such that $Q_m(\nu) \neq 0$ and $\lambda_1 \notin \operatorname{Span}(\mu, \nu)$, define $s_1(t), \ldots, s_m(t)$ as distinct real roots of $Q_m(\mu + s\nu + t\lambda_1)$ (see Theorem 6.3). Then

$$Q_m(\mu + s\nu + t\lambda_1) = Q_m(\nu) \prod_{i=1}^{m} (s - s_i(t)),$$

$$\frac{Q_{m-1}}{Q_m}(\mu + s\nu) = \frac{d}{dt} \log Q_m(\mu + s\nu + t\lambda_1)|_{t=0} = -\sum_{i=1}^{m} \frac{s_i'(0)}{s - s_i(0)},$$

$$\frac{d}{ds}\frac{Q_{m-1}}{Q_m}(\mu+s\nu)=\sum_{i=1}^{m}\frac{s_i'(0)}{(s-s_i(0))^2},\quad \frac{d^2}{ds^2}\frac{Q_{m-1}}{Q_m}(\mu+s\nu)=-2\sum_{i=1}^{m}\frac{s_i'(0)}{(s-s_i(0))^3},$$

Hence by using the formula $u^3(1/u)''=2(u')^2-uu''$, for $s=0$ we get

$$\frac{Q_{m-1}^3}{Q_m^3}(\mu)\frac{d^2}{ds^2}\frac{Q_m}{Q_{m-1}}(\mu+s\nu)=2\left(\sum_{i=1}^{m}\frac{s_i'(0)}{s_i^2(0)}\right)^2-2\sum_{i=1}^{m}\frac{s_i'(0)}{s_i(0)}\sum_{i=1}^{m}\frac{s_i'(0)}{s_i^3(0)}.$$

The latter is negative, which follows from the Cauchy inequality and the fact that $s_i'(0)s_i(0)>0$.

Thus, $(Q_m/Q_{m-1})(\mu+s\nu)$ as a function of s has negative second-order derivative at $s=0$. The same is true for $(Q_m/Q_{m-1})(\mu+s_0\nu+s\nu)$ with any s_0 provided $\mu+s_0\nu\in\Lambda_m$, and therefore, the function Q_m/Q_{m-1} is concave on that part of the straight line with direction ν passing through μ which lies in Λ_m. We have some restrictions on μ, ν, but they still allow us to take μ, ν everywhere dense (in corresponding sets), which along with the continuity of Q_m/Q_{m-1} proves its concavity in Λ_m.

Finally, the case $d=1$ is trivial and the case $d=2$ can, for example, formally be reduced to the case $d=3$ if in the very beginning we introduce a new independent variable λ^0 and define $\tilde{Q}_{m+1}(\lambda^0,\dots,\lambda^d)=\lambda^0Q_m(\lambda)$. The theorem is proved.

Corollary 6.5. *Define* $P_d(\lambda)=\lambda^1\cdot\dots\cdot\lambda^d$, *take* $\lambda_0=(1,\dots,1)$, *and for* $m=0,\dots,d$ *put*

$$P_m(\lambda)=\frac{m!}{d!}\frac{d^{d-m}}{dt^{d-m}}P_d(\lambda+t\lambda_0)|_{t=0},$$

so that $P_m(\lambda)$ *is the normalized* m*th elementary symmetric function. Then*

(i) *the polynomials* P_m *satisfy the hypotheses of Theorem* 6.4; *in particular, the function* $P_m^{1/m}$ *is concave in* Λ_m;

(ii) *for any* $k<m$ *and* $\lambda\in\partial\Lambda_m$ *the function* $P_kP_m^{-1}(\mu)$ *tends to infinity as* $\Lambda_m\ni\mu\to\lambda$.

Next, take a number $\alpha>0$ *and integers* $k\le r\le m$. *Then*

(iii) *the following functions are convex on* Λ_m:

$$P_k^{-\alpha},\quad (P_kP_r)^{-\alpha},\quad \left(\frac{P_k}{P_r}\right)^{\alpha},\quad \frac{P_kP_r}{P_m^2};$$

(iv) *for* $\lambda\in\Lambda_m$ *the function* $P_m^{1/m}P_k^{-1/k}(\lambda+t\lambda_0)$ *is an increasing function of* t *on* $[0,\infty)$, *and the function* $\log P_k(\lambda)$ *is a concave function of* k *for* $k=0,1,\dots,m$.

To prove (i), note that, obviously, P_d satisfies the hypotheses of Theorem 6.4 and by the same theorem (with $\lambda_1=\lambda_0$) the same is true for all P_m. Assertion (ii) is an immediate consequence of the theorem and of the formula $P_kP_m^{-1}=P_kP_{k+1}^{-1}\cdot\dots\cdot P_{m-1}P_m^{-1}$.

Assertion (iii) follows from the observation that on $\Lambda_m\subset\Lambda_r\subset\Lambda_k$ functions $-\log P_k$, $-\alpha\log P_k$, $P_k^{-\alpha}$, $-\alpha\log(P_kP_r)$, $(P_kP_r)^{-\alpha}$, $\log(P_{r-1}P_r^{-1})$,

$$\log(P_{r-2}P_r^{-1})=\log(P_{r-2}P_{r-1}^{-1})+\log(P_{r-1}P_r^{-1}),$$

$\alpha\log(P_kP_r^{-1})$, $P_k^{\alpha}P_r^{-\alpha}$, $\log(P_kP_m^{-1})+\log(P_rP_m^{-1})$ are all convex.

The first assertion in (iv) only needs to be proved for $k = m - 1$. But if we use the characterization of concave functions with the help of corresponding tangent planes, then we see that the concavity of P_m/P_{m-1} along with its homogeneity means that for any λ, $\mu \in \Lambda_m$ we have

$$\frac{P_m}{P_{m-1}}(\mu) \leq \frac{P_m}{P_{m-1}}(\lambda)\left[\frac{P_{m\lambda^i}}{P_m}(\lambda) - \frac{P_{m-1,\lambda^i}}{P_{m-1}}(\lambda)\right]\mu^i.$$

Here we put $\mu = \lambda_0$ and make use of the obvious relation $P_{m\lambda^i}\lambda_0^i = mP_{m-1}$. Then we see that for $\mu = \lambda_0$ the above inequality means that $P_mP_{m-2}(\lambda) \leq P_{m-1}^2(\lambda)$. We rediscover the Maclaurin inequality (in a particular case, the general case can be found in [10]). This inequality means that for $\lambda \in \Lambda_m$ the function $\log P_k(\lambda)$ is a concave function of k for $k = 0, 1, \dots, m$. To finish the proof of (iv) it remains to note that the same Maclaurin inequality means exactly that the derivative with respect to t of $\log[P_m^{1/m}P_{m-1}^{-1/(m-1)}(\lambda + t\lambda_0)]$ at the point $t = 0$ is positive.

Corollary 6.5 and Theorem 6.4 obviously imply the assertions of Lemmas 4.16 and 5.13 if we also make use of the following lemma from [21].

Lemma 6.6. *Let $G(\lambda) = G(\lambda^1, \dots, \lambda^d)$ be a convex function defined in a convex domain $S \subset \mathbb{R}^d$. Assume that G and S are invariant under all permutations of coordinates λ^i. Then the function $G(\lambda(w))$ is convex in the set $\{w : \lambda(w) \in S\}$, which is convex as well.*

Remark 6.7. With the help of Corollary 6.5 and Theorem 6.4 we can construct many different examples of fully nonlinear elliptic equations. For instance, take integers $m, r(0), \dots, r(m-1)$ such that $m \in [1, d]$, $m \geq r(k) > k$ for $k = 0, \dots, m-1$, and numbers $f \geq 0, c_0, \dots, c_{m-1} \geq 0$, $c_m > 0$, $\alpha_1, \dots, \alpha_m > 0$. Then any of the equations

$$P_m(u_{x^ix^i}) = f, \quad 1 = \sum_{k=0}^{m} c_k P_k^{-\alpha_k}(u_{x^ix^i}),$$

$$1 = \sum_{k=1}^{m} c_k (P_kP_{k-1})^{-\alpha_k}(u_{x^ix^i}), \quad 1 = \sum_{k=0}^{m-1} c_{k+1}\frac{P_k}{P_{r(k)}}(u_{x^ix^i}),$$

$$P_m^2(u_{x^ix^i}) = \sum_{k=0}^{m-1} c_kP_kP_{k+1}(u_{x^ix^i})$$

is an elliptic equation. Moreover, any of these equations has an elliptic branch described by the additional requirements that for $t > 0$ and $u_{x^ix^j} + t\delta_{ij}$ substituted instead of $u_{x^ix^j}$ its left-hand side is strictly greater than the right-hand side and $(u_{x^ix^j} + t\delta_{ij}) \in C_m$. Furthermore, these elliptic branches are equivalent to weakly nondegenerate Bellman equations with constant and bounded coefficients and with $\operatorname{tr} a(\omega) \equiv 1$. Finally, for the second, third, and fourth equations the corresponding Bellman equations are quasi-nondegenerate, and the same is true for the first and the last ones if in addition $f > 0$, $\sum_{k \leq m-1} c_k > 0$.

These statements can be proved in exactly the same way as Theorem 4.15. The so-called complex versions of our equations can be investigated along the same lines (cf. [4], [20]).

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